

ASYMPTOTIC ISOTROPIZATION IN INHOMOGENEOUS COSMOLOGY

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Abstract

In this paper we investigate asymptotic isotropization. We derive the asymptotic dynamics of spatially inhomogeneous cosmological models with a perfect fluid matter source and a positive cosmological constant near the de Sitter equilibrium state at late times, and near the flat FL equilibrium state at early times. Our results show that there exists an open set of solutions approaching the de Sitter state at late times, consistent with the cosmic no-hair conjecture. On the other hand, solutions that approach the flat FL state at early times are special and admit a so-called isotropic initial singularity. For both classes of models the asymptotic expansion of the line element contains an arbitrary spatial metric at leading order, indicating asymptotic spatial inhomogeneity. We show, however, that in the asymptotic regimes this spatial inhomogeneity is significant only at super-horizon scales.

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1 Introduction

In a recent paper we presented a general framework for analyzing the dynamics of generic spatially inhomogeneous cosmological models, referred to briefly as G_0 cosmologies (see Ref. [32], which we will henceforth refer to as Paper I). We employed Hubble-normalized scale-invariant variables which were defined within the orthonormal frame formalism, and led to the formulation of Einstein's field equations with a perfect fluid matter source as an autonomous system of evolution equations and constraints. In this paper we show that our framework can be used to derive asymptotic expansions for spatially inhomogeneous cosmologies that undergo asymptotic isotropization at late times or at the initial singularity.

It is part of the folklore of relativistic cosmology that all ever-expanding cosmological models with a positive cosmological constant asymptotically approach the de Sitter solution. A statement to this effect can be found in Gibbons and Hawking [10], p. 2739, in connection with a program to extend to cosmological event horizons certain ideas concerning thermodynamics and particle creation that had been applied to black hole event horizons. This conjecture, the so-called cosmic no-hair conjecture, was also made by Hawking and Moss [15], p. 36, after the introduction of the paradigm of cosmic inflation. Shortly thereafter, Starobinskii [30] took the first steps in describing the asymptotic structure of the metric for spatially inhomogeneous cosmologies with a perfect fluid matter source and an effective cosmological constant that approach the de Sitter solution at late times, showing that the line element contained 8 (the maximum possible number) free functions of the spatial coordinates. He commented that the asymptotic solution described “exponentially rapid local isotropization of the universe”, and coined the phrase “the cosmological constant is the best ‘isotropizer’”. In the same year, Wald [33] gave a proof of the conjecture for ever-expanding spatially homogeneous (SH) cosmologies. Subsequently

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Jensen and Stein-Schabes [17] extended Wald's proof to spatially inhomogeneous cosmologies, subject to the restriction that the 3-Ricci curvature scalar is negative. The significance of this result is not clear, however, for the following reason. The complete set of SH cosmologies can be divided into one invariant subset for which the 3-Ricci curvature scalar is never positive (Bianchi Type-I to Type-VIII) for which Wald's theorem applies, and another invariant subset containing Bianchi Type-IX and Kantowski-Sachs models, for which the dynamical behavior of the 3-Ricci curvature scalar is more complicated (some models approach the de Sitter solution while others re-collapse). Such a natural division into invariant subsets does not exist in the G_0 case. Hence, in G_0 cosmology the situation is still unclear as regards possible "cosmic no-hair theorems". A variant of such a theorem, however, was proved very recently by Tchapnda and Rendall [31] for cosmologies with collisionless matter and positive cosmological constant that are either plane or hyperbolic-plane symmetrical.

The motivation for much of the above work was provided by the idea of cosmic inflation, in that the de Sitter solution is regarded as describing the Universe in the late stages of an inflationary epoch. On the other hand, the de Sitter solution can also be regarded as a simple prototype of a cosmological model undergoing accelerated expansion during the present epoch. Indeed, recent observations have focused attention on such models. In particular, five years ago two independently working observational groups reported that the analysis of the type Ia supernova redshift data they had gathered suggested that in the present epoch the Universe may be in a state of accelerated expansion (see, e.g., Perlmutter *et al* [23], Schmidt *et al* [28], and Schmidt [27]). The acceleration could be due to either a veritable, repulsive cosmological constant, or some form of "dark energy"; see Carroll [6] and Ellis [9] for further details and references.

In this paper, motivated by the above discussion, we consider G_0 cosmologies that are future asymptotic to the de Sitter solution. We give a formal definition of this notion within the Hubble-normalized state space and then derive asymptotic expansions, valid as $t \rightarrow \infty$, for the Hubble-normalized variables.

The second type of asymptotic isotropization, namely isotropization at the initial singularity, arises in connection with the notion of *quiescent cosmology* (cf., e.g., Barrow [2]), which provides an alternative to cosmic inflation. The idea is that, due to entropy considerations on a cosmological scale, a suitable initial condition for the Universe is that the Weyl curvature should be zero (or at least dynamically unimportant) at the initial singularity; this is the *Weyl curvature hypothesis* by Penrose [22], p. 630. This hypothesis leads to the notion of an *isotropic initial singularity*. Cosmological initial singularities of this type first arose as a special case in the general analysis of initial singularities performed by Lifshitz and Khalatnikov (LK hereafter) [19], p. 203. This type of initial singularity was also encountered in the work of Eardley, Liang and Sachs [8], p. 101. Subsequently, Goode and Wainwright [13] gave a formal definition of an isotropic initial singularity, using a conformally related metric, and derived various properties. Motivated by these ideas, we consider G_0 cosmologies that are past asymptotic to the flat FL solution and give a formal definition of this notion within the Hubble-normalized state space and then derive asymptotic expansions, valid as $t \rightarrow -\infty$, for the Hubble-normalized variables.

In this paper, we also extend the formalism of Paper I to cover geodesic null congruences and structures in connection with the issue of the formation of particle or event horizons (cf. the classic paper by Rindler [26]), thus naturally generalizing the work by Nilsson *et al* [21] from an SH context to a general spatially inhomogeneous setting.

The plan of this paper is as follows. In Sec. 2, we present the Hubble-normalized evolution equations and constraints for G_0 cosmologies that arise from Einstein's field equations and the matter equations. In Sec. 3, we give the detailed asymptotic form of the Hubble-normalized variables for G_0 cosmologies that approach the de Sitter solution at late times, and discuss various features of this class of cosmological models. In Sec. 4, we present analogous results for G_0 cosmologies that approach the flat FL solution at early times. We conclude in Sec. 5 with a discussion of the analogies between the two classes of cosmological models that undergo asymptotic isotropization, and raise some issues for future study. The proofs of the results are given in the appendix.

2 Evolution equations and constraints

We consider spatially inhomogeneous cosmological models with a positive cosmological constant, Λ , and a perfect fluid matter source with a linear barotropic equation of state. We thus have

$$\tilde{p}(\tilde{\mu}) = (\gamma - 1)\tilde{\mu}, \quad (1)$$

where $\tilde{\mu}$ is the total energy density of the fluid (assumed to be non-negative) and \tilde{p} its isotropic pressure, in the rest 3-spaces associated with the fluid 4-velocity vector field $\tilde{\mathbf{u}}$, while γ is a constant parameter. The range

$$1 \leq \gamma < 2 \quad (2)$$

is of particular physical interest, since it ensures that the perfect fluid satisfies the dominant and strong energy conditions and the causality requirement that the speed of sound should be less than that of light. The values $\gamma = 1$ and $\gamma = \frac{4}{3}$ correspond to pressure-free matter (“dust”) and incoherent radiation, respectively.

We express the fluid 4-velocity vector field by

$$\tilde{\mathbf{u}} := \Gamma(\mathbf{e}_0 + \mathbf{v}) , \quad (3)$$

where \mathbf{e}_0 is a vorticity-free timelike reference congruence, \mathbf{v} the peculiar velocity vector field of the fluid relative to the rest 3-spaces of \mathbf{e}_0 , and

$$\Gamma := \frac{1}{\sqrt{1 - v^2}} , \quad v^2 := v_a v^a , \quad (4)$$

defines the usual Lorentz factor. To obtain an orthonormal frame, $\{\mathbf{e}_a\}_{a=0,1,2,3}$, we supplement the vector field \mathbf{e}_0 with an orthonormal spatial frame $\{\mathbf{e}_\alpha\}_{\alpha=1,2,3}$ in the rest 3-spaces of \mathbf{e}_0 . The frame metric is then given by $\eta_{ab} = \text{diag}[-1, 1, 1, 1]$. To convert the dynamical equations of the orthonormal frame formalism to a system of partial differential equations, it is necessary to introduce a set of local coordinates $\{x^\mu\}_{\mu=0,1,2,3} = \{t, x^i\}_{i=1,2,3}$.

2.1 Hubble-normalized Einstein–Euler equations in separable volume gauge

In Paper I, we employed the orthonormal frame formalism and introduced Hubble-normalized scale-invariant frame, connection and matter variables, as follows.¹

$$\{\boldsymbol{\partial}_0, \boldsymbol{\partial}_\alpha\} := \{\mathbf{e}_0, \mathbf{e}_\alpha\}/H \quad (5)$$

$$\{\dot{U}^\alpha, A^\alpha, N_{\alpha\beta}, \Sigma_{\alpha\beta}, R^\alpha\} := \{\dot{u}^\alpha, a^\alpha, n_{\alpha\beta}, \sigma_{\alpha\beta}, \Omega^\alpha\}/H \quad (6)$$

$$\{\Omega, \Omega_\Lambda\} := \{\mu, \Lambda\}/(3H^2) . \quad (7)$$

Expressing the Hubble-normalized frame vector fields with respect to a local coordinate basis leads to:

$$\boldsymbol{\partial}_0 = \mathcal{N}^{-1} (\partial_t - N^i \partial_i) , \quad \boldsymbol{\partial}_\alpha = E_\alpha{}^i \partial_i , \quad (8)$$

where \mathcal{N} and N^i are viewed as coordinate gauge source functions, while the $E_\alpha{}^i$ are dependent variables that we refer to as the frame variables.

The introduction of Hubble-normalized scale-invariant variables leaves H as the only dependent variable carrying a physical dimension, namely $[\text{length}]^{-1}$. The evolution equation for H decouples and is given by

$$\boldsymbol{\partial}_0 H = -(q + 1) H , \quad (9)$$

where q is the deceleration parameter, familiar from observational cosmology. The expression for q , which depends on the choice of temporal gauge, is given below. It is also necessary to introduce the spatial Hubble gradient r_α , defined by

$$r_\alpha := -\frac{1}{H} \boldsymbol{\partial}_\alpha H . \quad (10)$$

Throughout this paper, we will employ the *separable volume gauge* introduced in Paper I, which is characterized by the following choice of coordinate and frame gauge source functions:

$$N^i = 0 , \quad \mathcal{N} = 1 \quad \Rightarrow \quad \dot{U}_\alpha = r_\alpha ; \quad (11)$$

¹We use units such that Newton’s gravitational constant G and the speed of light in vacuum c are given by $8\pi G/c^2 = 1$ and $c = 1$. Then frame and connection variables have physical dimension $[\text{length}]^{-1}$, while curvature variables have physical dimension $[\text{length}]^{-2}$.

the choice $\mathcal{N} = 1$ corresponds to setting the dimensional lapse function equal to the momentary Hubble radius, $N = H^{-1}$. With this choice of temporal gauge the deceleration parameter is given by

$$q = 2\Sigma^2 + \frac{1}{2} G_+^{-1} [(3\gamma - 2) + (2 - \gamma)v^2] \Omega - \Omega_\Lambda - \frac{1}{3} (\boldsymbol{\partial}_\alpha - 2A_\alpha) r^\alpha, \quad (12)$$

where $\Sigma^2 = \frac{1}{6} (\Sigma_{\alpha\beta} \Sigma^{\alpha\beta})$, and the scalars G_\pm are defined by (see Eq. (6) in Paper I)

$$G_\pm = 1 \pm (\gamma - 1)v^2. \quad (13)$$

It should be noted that the temporal gauge choice (11) does not determine a unique family of spacelike 3-surfaces $\mathcal{S}:\{t = \text{constant}\}$ and associated normal timelike reference congruence \mathbf{e}_0 . In Sec. 4 [see Eq. (125)], we will use this remaining freedom to simplify the asymptotic expansions that we derive. There is also freedom in the choice of the spatial frame vectors and spatial coordinates. For the applications in this paper it is desirable to choose a Fermi-propagated spatial frame, i.e., one which satisfies

$$R^\alpha = 0. \quad (14)$$

The remaining spatial gauge freedom is a time-independent rotation of the spatial frame vectors,

$$\tilde{\mathbf{e}}_\alpha = \mathcal{O}_\alpha{}^\beta(x^i) \mathbf{e}_\beta. \quad (15)$$

There is also the following freedom in the choice of spatial coordinates:

$$\tilde{x}^i = f^i(x^j). \quad (16)$$

In the separable volume gauge with Fermi-propagated spatial frame, the Hubble-normalized state vector for G_0 cosmologies is given by

$$\mathbf{X} = (E_\alpha{}^i, r_\alpha, A^\alpha, N_{\alpha\beta}, \Sigma_{\alpha\beta}, \Omega, v^\alpha, \Omega_\Lambda)^T. \quad (17)$$

The evolution equations and constraints that the components of \mathbf{X} have to satisfy are given below.²

Evolution equations:

$$\partial_t E_\alpha{}^i = (q \delta_\alpha{}^\beta - \Sigma_\alpha{}^\beta + \epsilon_{\alpha\gamma}{}^\beta R^\gamma) E_\beta{}^i \quad (18)$$

$$\partial_t r_\alpha = (q \delta_\alpha{}^\beta - \Sigma_\alpha{}^\beta + \epsilon_{\alpha\gamma}{}^\beta R^\gamma) r_\beta + \boldsymbol{\partial}_\alpha q \quad (19)$$

$$\partial_t A^\alpha = (q \delta^\alpha{}_\beta - \Sigma^\alpha{}_\beta + \epsilon^\alpha{}_{\gamma\beta} R^\gamma) A^\beta + \frac{1}{2} \boldsymbol{\partial}_\beta (\Sigma^{\alpha\beta} + \epsilon^{\alpha\beta}{}_\gamma R^\gamma) \quad (20)$$

$$\begin{aligned} \partial_t \Sigma^{\alpha\beta} = & (q - 2) \Sigma^{\alpha\beta} - 2N^{\langle\alpha}{}_\gamma N^{\beta\rangle\gamma} + N_\gamma{}^\gamma N^{\langle\alpha\beta\rangle} - \delta^{\gamma\langle\alpha} (\boldsymbol{\partial}_\gamma - r_\gamma) A^{\beta\rangle} \\ & + \epsilon^{\gamma\delta\langle\alpha} [(\boldsymbol{\partial}_\gamma - 2A_\gamma) N^{\beta\rangle\delta} + 2R_\gamma \Sigma^{\beta\rangle\delta}] + (\delta^{\gamma\langle\alpha} \boldsymbol{\partial}_\gamma + A^{\langle\alpha} r^{\beta\rangle}) + 3 \frac{\gamma}{G_+} \Omega v^{\langle\alpha} v^{\beta\rangle} \end{aligned} \quad (21)$$

$$\partial_t N^{\alpha\beta} = (q \delta^{\langle\alpha}{}_\delta + 2\Sigma^{\langle\alpha}{}_\delta + 2\epsilon^{\gamma\delta\langle\alpha} R_\gamma) N^{\beta\rangle\delta} - \boldsymbol{\partial}_\gamma (\epsilon^{\gamma\delta\langle\alpha} \Sigma^{\beta\rangle\delta} - \delta^{\gamma\langle\alpha} R^{\beta\rangle}) + \delta^{\alpha\beta} R^\gamma \quad (22)$$

$$\begin{aligned} \partial_t \Omega = & -\frac{\gamma}{G_+} v^\alpha \boldsymbol{\partial}_\alpha \Omega + G_+^{-1} [2G_+ q - (3\gamma - 2) - (2 - \gamma)v^2 - \gamma (\Sigma_{\alpha\beta} v^\alpha v^\beta) \\ & - \gamma (\boldsymbol{\partial}_\alpha - 2A_\alpha) v^\alpha + \gamma v^\alpha \boldsymbol{\partial}_\alpha \ln G_+] \Omega \end{aligned} \quad (23)$$

$$\begin{aligned} \partial_t v^\alpha = & -v^\beta \boldsymbol{\partial}_\beta v^\alpha + \delta^{\alpha\beta} \boldsymbol{\partial}_\beta \ln G_+ - \frac{(\gamma - 1)}{\gamma} (1 - v^2) \delta^{\alpha\beta} (\boldsymbol{\partial}_\beta \ln \Omega - 2r_\beta) \\ & + G_+^{-1} \left[(\gamma - 1) (1 - v^2) (\boldsymbol{\partial}_\beta v^\beta) - (2 - \gamma) v^\beta \boldsymbol{\partial}_\beta \ln G_+ \right. \\ & + \frac{(\gamma - 1)}{\gamma} (2 - \gamma) (1 - v^2) v^\beta (\boldsymbol{\partial}_\beta \ln \Omega - 2r_\beta) + (3\gamma - 4) (1 - v^2) \\ & + (2 - \gamma) (\Sigma_{\beta\gamma} v^\beta v^\gamma) + G_- (r_\beta v^\beta) + [G_+ - 2(\gamma - 1)] (A_\beta v^\beta) \left. \right] v^\alpha \\ & - \Sigma^\alpha{}_\beta v^\beta + \epsilon^\alpha{}_{\beta\gamma} R^\beta v^\gamma - r^\alpha - v^2 A^\alpha + \epsilon^{\alpha\beta\gamma} N_{\beta\delta} v_\gamma v^\delta \end{aligned} \quad (24)$$

$$\partial_t \Omega_\Lambda = 2(q + 1) \Omega_\Lambda. \quad (25)$$

²These equations are obtained by imposing the restriction (11) on Eqs. (33)–(35), (38), (44), (47), (144) and (145) in Paper I. We keep the R^α in the equations for future reference.

where q , G_{\pm} and $\boldsymbol{\theta}_{\alpha}$ are given by Eqs. (12), (13) and (8).

Constraints:

$$0 = (\mathcal{C}_{\text{com}})^i_{\alpha\beta} := 2(\boldsymbol{\theta}_{[\alpha} - r_{[\alpha} - A_{[\alpha} E_{\beta]}]^i - \epsilon_{\alpha\beta\delta} N^{\delta\gamma} E_{\gamma}^i \quad (26)$$

$$0 = (\mathcal{C}_G) := 1 - \Omega_k - \Sigma^2 - \Omega - \Omega_{\Lambda} \quad (27)$$

$$0 = (\mathcal{C}_C)^{\alpha} := \boldsymbol{\theta}_{\beta} \Sigma^{\alpha\beta} + (2\delta^{\alpha}_{\beta} - \Sigma^{\alpha}_{\beta}) r^{\beta} - 3A_{\beta} \Sigma^{\alpha\beta} - \epsilon^{\alpha\beta\gamma} N_{\beta\delta} \Sigma_{\gamma}^{\delta} + 3 \frac{\gamma}{G_{+}} \Omega v^{\alpha} \quad (28)$$

$$0 = (\mathcal{C}_J)^{\alpha} := (\boldsymbol{\theta}_{\beta} - r_{\beta}) (N^{\alpha\beta} + \epsilon^{\alpha\beta\gamma} A_{\gamma}) - 2A_{\beta} N^{\alpha\beta} \quad (29)$$

$$0 = (\mathcal{C}_{\Lambda})_{\alpha} := (\boldsymbol{\theta}_{\alpha} - 2r_{\alpha}) \Omega_{\Lambda} . \quad (30)$$

Auxiliary 3-curvature variables:

$$\Omega_k := -\frac{1}{3} (2\boldsymbol{\theta}_{\alpha} - 2r_{\alpha} - 3A_{\alpha}) A^{\alpha} + \frac{1}{6} (N_{\alpha\beta} N^{\alpha\beta}) - \frac{1}{12} (N_{\alpha}^{\alpha})^2 \quad (31)$$

$$\mathcal{S}_{\alpha\beta} := -\frac{1}{3} \epsilon^{\gamma\delta}_{\langle\alpha} (\boldsymbol{\theta}_{|\gamma|} - r_{|\gamma|} - 2A_{|\gamma|}) N_{\beta\rangle\delta} + \frac{1}{3} (\boldsymbol{\theta}_{\langle\alpha} - r_{\langle\alpha}) A_{\beta\rangle} + \frac{2}{3} N_{\langle\alpha}^{\gamma} N_{\beta\rangle\gamma} - \frac{1}{3} N_{\gamma}^{\gamma} N_{\langle\alpha\beta\rangle} \quad (32)$$

$$\mathcal{C}_{\alpha\beta} := \epsilon^{\gamma\delta}_{\langle\alpha} (\boldsymbol{\theta}_{|\gamma|} - 2r_{|\gamma|} - A_{|\gamma|}) \mathcal{S}_{\beta\rangle\delta} - 3N_{\langle\alpha}^{\gamma} \mathcal{S}_{\beta\rangle\gamma} + \frac{1}{2} N_{\gamma}^{\gamma} \mathcal{S}_{\alpha\beta} , \quad (33)$$

where $\mathcal{S}_{\alpha\beta}$ and Ω_k are the tracefree part and trace part of the Hubble-normalized 3-Ricci curvature of a spacelike 3-surface $\mathcal{S}:\{t = \text{constant}\}$, respectively, while $\mathcal{C}_{\alpha\beta}$ is the Hubble-normalized 3-Cotton–York tensor which provides information on the conformal curvature properties of a spacelike 3-surface $\mathcal{S}:\{t = \text{constant}\}$.

The 3-Ricci curvature variables $\mathcal{S}_{\alpha\beta}$ and Ω_k satisfy the Hubble-normalized twice-contracted 3-Bianchi identity, given by

$$0 \equiv (\boldsymbol{\theta}_{\beta} - 2r_{\beta} - 3A_{\beta}) \mathcal{S}^{\alpha\beta} - \epsilon^{\alpha\beta\gamma} N_{\beta\delta} \mathcal{S}_{\gamma}^{\delta} + \frac{1}{3} \delta^{\alpha\beta} (\boldsymbol{\theta}_{\beta} - 2r_{\beta}) \Omega_k . \quad (34)$$

2.2 Energy-normalized null geodesics augmentation

A few years ago, Nilsson *et al* [21] extended the Hubble-normalized equations for SH cosmology by adding the equations for an energy-normalized geodesic (timelike or null) congruence. This strategy turned out to be very useful, both from an analytical and a numerical point of view. We therefore generalize that approach to the present G_0 cosmology setting, and this will subsequently allow us to investigate the connection between asymptotic silence of the gravitational field dynamics in relativistic cosmology and the formation of particle or event horizons (see Paper I, Sec. 4.1).

Instead of being interested in the null geodesics emanating from a single given event, we will be interested in all possible null geodesics. We will thus not consider the problem of integrating a system of ordinary differential equations (associated with the former situation), but will instead consider all possible null geodesic vector fields (which, at a later stage, can be integrated to yield all possible associated flows). We therefore start by considering a vector field k^a which is tangent to a geodesic null congruence. This vector field is thus governed by the equations

$$k^b \nabla_b k^a = 0 \quad (35)$$

$$k_a k^a = 0 . \quad (36)$$

We now make a (3+1)-split of Eq. (35) with respect to a general orthonormal frame. Using the notation of Paper I and Ref. [21], this yields

$$\mathcal{E} e_0(\mathcal{E}) = -k^{\alpha} (e_{\alpha} + \dot{u}_{\alpha}) (\mathcal{E}) - (H \delta_{\alpha\beta} + \sigma_{\alpha\beta}) k^{\alpha} k^{\beta} \quad (37)$$

$$\begin{aligned} \mathcal{E} e_0(k^{\alpha}) &= -k^{\beta} e_{\beta}(k^{\alpha}) - \mathcal{E} (H \delta^{\alpha}_{\beta} + \sigma^{\alpha}_{\beta}) k^{\beta} - (k_{\beta} k^{\beta}) a^{\alpha} + (k^{\beta} a_{\beta}) k^{\alpha} - \epsilon^{\alpha}_{\beta\delta} n_{\gamma}^{\delta} k^{\beta} k^{\gamma} \\ &\quad - \mathcal{E}^2 \dot{u}^{\alpha} + \mathcal{E} \epsilon^{\alpha}_{\beta\gamma} \Omega^{\beta} k^{\gamma} , \end{aligned} \quad (38)$$

while a (3+1)-split of Eq. (36) yields

$$-\mathcal{E}^2 + (k_{\alpha} k^{\alpha}) = 0 . \quad (39)$$

Here $\mathcal{E} := k^0$ denotes the energy (frequency) of the particles that are traveling along the geodesic null congruence.

We now employ the Hubble-normalized gravitational field variables given in Eqs. (5) and (6), and in addition introduce *energy-normalized* null vector variables as in Ref. [21] according to

$$K^\alpha := \frac{k^\alpha}{\mathcal{E}} . \quad (40)$$

With Eq. (39) we thus find

$$K_\alpha K^\alpha = 1 , \quad (41)$$

where the K^α correspond to the direction cosines of the null geodesics.

Energy-normalization implies that the dimensional equation (37) for \mathcal{E} decouples and can be written as

$$\partial_0 \mathcal{E} = -(s+1) \mathcal{E} , \quad (42)$$

where s is given by

$$s := (\Sigma_{\alpha\beta} K^\alpha K^\beta) - (t_\alpha - \dot{U}_\alpha) K^\alpha , \quad (43)$$

which is obtained from Eqs. (37), (40), (41) and (6). It is also necessary to introduce the *spatial energy gradient* t_α , defined by

$$t_\alpha := -\frac{1}{\mathcal{E}} \partial_\alpha \mathcal{E} , \quad (44)$$

and governed by

$$\partial_0 t_\alpha = (q \delta_\alpha^\beta - \Sigma_\alpha^\beta + \epsilon_{\alpha\gamma}{}^\beta R^\gamma) t_\beta + (\partial_\alpha - r_\alpha + \dot{U}_\alpha) (s+1) \quad (45)$$

$$0 = (\mathcal{C}_t)^\alpha := [\epsilon^{\alpha\beta\gamma} (\partial_\beta - r_\beta - A_\beta) - N^{\alpha\gamma}] t_\gamma . \quad (46)$$

The latter equations are obtained by choosing $f = \mathcal{E}$ in the commutator equations (31) and (32) of Paper I, and making use of Eqs. (42) and (44) given above. Equations (45) and (46) constitute integrability conditions for Eqs. (42) and (44).

We now write the evolution equation (38) in energy/Hubble-normalized form.

$$\begin{aligned} \partial_0 K^\alpha &= -K^\beta (\partial_\beta - t_\beta - A_\beta) K^\alpha + (s \delta^\alpha_\beta - \Sigma^\alpha_\beta) K^\beta - A^\alpha - \epsilon^\alpha_{\beta\delta} N_\gamma{}^\delta K^\beta K^\gamma \\ &\quad - \dot{U}^\alpha + \epsilon^\alpha_{\beta\gamma} R^\beta K^\gamma . \end{aligned} \quad (47)$$

Note that the contraction of Eq. (47) with K_α vanishes identically on account of Eqs. (41) and (43).

Let us now consider these equations in the separable volume gauge, specified by Eqs. (11). Then

$$\begin{aligned} \partial_t K^\alpha &= -K^\beta (\partial_\beta - t_\beta - A_\beta) K^\alpha + (s \delta^\alpha_\beta - \Sigma^\alpha_\beta) K^\beta - A^\alpha - \epsilon^\alpha_{\beta\delta} N_\gamma{}^\delta K^\beta K^\gamma \\ &\quad - r^\alpha + \epsilon^\alpha_{\beta\gamma} R^\beta K^\gamma \end{aligned} \quad (48)$$

$$\partial_t t_\alpha = (q \delta_\alpha^\beta - \Sigma_\alpha^\beta + \epsilon_{\alpha\gamma}{}^\beta R^\gamma) t_\beta + \partial_\alpha s , \quad (49)$$

where

$$s = (\Sigma_{\alpha\beta} K^\alpha K^\beta) - (t_\alpha - r_\alpha) K^\alpha . \quad (50)$$

Once the energy-normalized null vector variables K^α have been obtained, one can solve for the associated geodesic null congruence by integrating the relation

$$\frac{dx^i}{dt} = E_\alpha{}^i K^\alpha , \quad (51)$$

subject to appropriate initial conditions.

In this paper, we will use the above equations to determine asymptotic geodesic null structures and derive properties concerning particle and event horizons which are associated with asymptotic silence.

3 de Sitter-like future asymptotics

In this section we give the detailed asymptotic form of the Hubble-normalized variables for G_0 cosmologies that approach the de Sitter solution at late times.

3.1 G_0 cosmologies future asymptotic to the de Sitter solution

The de Sitter solution of Einstein's field equations describes a cosmological model with zero matter energy density and positive cosmological constant, which is undergoing exponential expansion. Employing a (3+1)-decomposition with intrinsically flat spacelike 3-surfaces, the line element and the associated timelike reference congruence are given by

$$ds^2 = -dT^2 + \ell_0^2 e^{2\sqrt{\Lambda/3}T} (dx^2 + dy^2 + dz^2) , \quad (52)$$

$$e_0 = \partial_T , \quad (53)$$

where $\Lambda > 0$ is the cosmological constant, and T is clock time. It is convenient to choose the unit of [length] as follows:

$$\ell_0 = \sqrt{\frac{3}{\Lambda}} . \quad (54)$$

The Hubble scalar is constant and positive, and is given by

$$H = \sqrt{\frac{\Lambda}{3}} = \ell_0^{-1} . \quad (55)$$

It follows from Eq. (9) that the deceleration parameter is constant and negative, being given by

$$q = -1 . \quad (56)$$

We also find it convenient to use a conformal time coordinate η , defined by

$$\eta := e^{-T/\ell_0} . \quad (57)$$

The line element is then cast into the form

$$ds^2 = \ell_0^2 \eta^{-2} (-d\eta^2 + dx^2 + dy^2 + dz^2) . \quad (58)$$

Relative to the natural orthonormal frame associated with the line element (52), the Hubble-normalized variables are

$$E_\alpha{}^i = e^{-t} \delta_\alpha{}^i , \quad r_\alpha = 0 , \quad (59)$$

$$\Sigma_{\alpha\beta} = 0 , \quad A^\alpha = 0 , \quad N_{\alpha\beta} = 0 , \quad (60)$$

$$\Omega = 0 , \quad \Omega_\Lambda = 1 , \quad (61)$$

where the dynamical time coordinate t is introduced via

$$t = \frac{T}{\ell_0} , \quad (62)$$

or, equivalently,

$$\eta = e^{-t} . \quad (63)$$

We note that the volume density \mathcal{V} is given by

$$\mathcal{V} = \ell_0^3 e^{3t} , \quad (64)$$

so that the separable volume gauge conditions are satisfied.

Observe that the frame variables satisfy $\lim_{t \rightarrow \infty} E_\alpha{}^i = 0$. Thus, the de Sitter solution is described by an orbit in the Hubble-normalized state space that is future asymptotic to an equilibrium point on the silent boundary, $E_\alpha{}^i = 0$.

Motivated by Eqs. (59)–(61), we say that a G_0 cosmology is future asymptotic to the de Sitter solution if the following limits are satisfied:

$$\lim_{t \rightarrow \infty} (E_\alpha{}^i, r_\alpha)^T = \mathbf{0} \quad (65)$$

$$\lim_{t \rightarrow \infty} (\Sigma_{\alpha\beta}, A^\alpha, N_{\alpha\beta})^T = \mathbf{0} \quad (66)$$

$$\lim_{t \rightarrow \infty} \Omega = 0 , \quad \lim_{t \rightarrow \infty} \Omega_\Lambda = 1 . \quad (67)$$

We note that for the de Sitter solution the peculiar velocity variables v^α are unrestricted, since the Codacci constraint (28) (which may be viewed as determining the v^α algebraically provided that $\Omega \neq 0$) yields no information. We will show that if the limits (65)–(67) are satisfied, together with certain technical restrictions, then the evolution equations determine the asymptotic form of the v^α as $t \rightarrow \infty$. Wald’s theorem [33], mentioned in the introduction, asserts that any SH cosmology with a positive cosmological constant, except those of Kantowski-Sachs and Bianchi Type-IX which may re-collapse, is future asymptotic to the de Sitter solution, in the above sense. As discussed in the introduction, at present it is not known whether Wald’s theorem can be generalized to G_0 cosmologies. We now consider the class of G_0 cosmologies which are future asymptotic to the de Sitter solution, and present the asymptotic form of the Hubble-normalized variables as $t \rightarrow \infty$.

3.2 Asymptotic expansions

In App. B, we prove that the de Sitter solution is asymptotically stable, and derive the asymptotic form of the Hubble-normalized variables for G_0 cosmologies that are future asymptotic to the de Sitter solution, in the sense of Eqs. (65)–(67). It is necessary to impose certain restrictions on the spatial derivatives; these are given in App. B.

We now give the asymptotic expansions, using the convention that “hatted” coefficients depend only on the spatial coordinates:

$$(E_\alpha{}^i, A^\alpha, N^{\alpha\beta})^T = (\hat{E}_\alpha{}^i, \hat{A}^\alpha, \hat{N}^{\alpha\beta})^T e^{-t} + \mathcal{O}(e^{-3t}) \quad (68)$$

$$r_\alpha = -\frac{1}{2} (\hat{E}_\alpha{}^i \partial_i \hat{\Omega}_k) e^{-3t} + \mathcal{O}(e^{-(1+3\gamma)t} + e^{-5t}) \quad (69)$$

$$\Sigma^{\alpha\beta} = -3\hat{\mathcal{S}}^{\alpha\beta} e^{-2t} + \hat{\Sigma}^{\alpha\beta} e^{-3t} + \mathcal{O}(e^{-4t}) \quad (70)$$

$$\Omega_\Lambda = 1 - \hat{\Omega}_k e^{-2t} + \mathcal{O}(e^{-3\gamma t} + e^{-4t}) \quad (71)$$

$$\Omega = \begin{cases} \hat{\Omega} e^{-3\gamma t} + \mathcal{O}(e^{(3\gamma-8)t}) & \text{for } 1 \leq \gamma < \frac{4}{3} \\ \hat{\Omega} e^{-4t} + \mathcal{O}(e^{-5t}) & \text{for } \gamma = \frac{4}{3} \\ \hat{\Omega} e^{-4t} + \mathcal{O}(e^{-5t} + e^{(-4-2\frac{(3\gamma-4)}{(2-\gamma))t})} & \text{for } \frac{4}{3} < \gamma < 2 \end{cases} \quad (72)$$

$$v^\alpha = \begin{cases} \hat{v}^\alpha e^{-t} + \mathcal{O}(e^{-3t}) & \text{for } \gamma = 1 \\ \hat{v}^\alpha e^{(3\gamma-4)t} + \mathcal{O}(e^{-t} + e^{3(3\gamma-4)t}) & \text{for } 1 < \gamma < \frac{4}{3} \\ \hat{v}^\alpha + \mathcal{O}(e^{-t}) & \text{for } \gamma = \frac{4}{3} \end{cases} \quad (73)$$

$$1 - v^2 = \exp \left[-2 \frac{(3\gamma-4)}{(2-\gamma)} t \right] \left[(1 - \hat{v}^2) + \mathcal{O}(e^{-t} + e^{-2\frac{(3\gamma-4)}{(2-\gamma)}t}) \right] \quad \text{for } \frac{4}{3} < \gamma < 2. \quad (74)$$

The coefficients \hat{A}^α and $\hat{N}^{\alpha\beta}$ are determined by $\hat{E}_\alpha{}^i$ according to

$$\hat{A}_\alpha = \hat{E}^\beta{}_i \hat{E}_{[\alpha}{}^j \partial_j \hat{E}_{\beta]}{}^i \quad (75)$$

$$\hat{N}^{\alpha\beta} = \hat{E}^{(\alpha}{}_{\epsilon} \epsilon^{\beta)\gamma\delta} \hat{E}_{\gamma}{}^j \partial_j \hat{E}_{\delta}{}^i, \quad (76)$$

where the matrix $\hat{E}^\alpha{}_i$ is the inverse of $\hat{E}_\alpha{}^i$:

$$\hat{E}^\alpha{}_i \hat{E}_\beta{}^i = \delta^\alpha{}_\beta, \quad \hat{E}^\alpha{}_i \hat{E}_\alpha{}^j = \delta_i{}^j. \quad (77)$$

The coefficients $\hat{\Omega}_k$ and $\hat{\mathcal{S}}_{\alpha\beta}$ in Eqs. (69)–(71) are the leading order coefficients in the asymptotic expansions of the 3-Ricci curvature variables,

$$\Omega_k = e^{-2t} \left[\hat{\Omega}_k + \mathcal{O}(e^{-2t}) \right] \quad (78)$$

$$\mathcal{S}_{\alpha\beta} = e^{-2t} \left[\hat{\mathcal{S}}_{\alpha\beta} + \mathcal{O}(e^{-2t}) \right]. \quad (79)$$

They are determined by $\hat{E}_\alpha{}^i$, \hat{A}^α and $\hat{N}_{\alpha\beta}$ according to:³

$$\hat{\Omega}_k = -\frac{1}{3} (2\hat{E}_\alpha{}^i \partial_i - 3\hat{A}_\alpha) \hat{A}^\alpha + \frac{1}{6} (\hat{N}_{\alpha\beta} \hat{N}^{\alpha\beta}) - \frac{1}{12} (\hat{N}_\alpha{}^\alpha)^2 \quad (80)$$

$$\hat{\mathcal{S}}_{\alpha\beta} = -\frac{1}{3} \epsilon^{\gamma\delta}{}_{\langle\alpha} (\hat{E}_{|\gamma|}{}^i \partial_i - 2\hat{A}_{|\gamma|}) \hat{N}_{\beta\rangle\delta} + \frac{1}{3} \hat{E}_{\langle\alpha}{}^i \partial_i \hat{A}_{\beta\rangle} + \frac{2}{3} \hat{N}_{\langle\alpha}{}^\gamma \hat{N}_{\beta\rangle\gamma} - \frac{1}{3} \hat{N}_\gamma{}^\gamma \hat{N}_{\langle\alpha\beta\rangle}. \quad (81)$$

³These equations are the “hatted” versions of Eqs. (31) and (32), with r_α set to zero.

The constraint $(C_C)^\alpha$ at order e^{-4t} provides the restriction for \hat{v}^α :

$$0 = (\hat{E}_\beta{}^i \partial_i - 3\hat{A}_\beta) \hat{\Sigma}^{\alpha\beta} - \epsilon^{\alpha\beta\gamma} \hat{N}_{\beta\delta} \hat{\Sigma}_\gamma{}^\delta + C^\alpha + \hat{Q}^\alpha, \quad (82)$$

where

$$C^\alpha = \begin{cases} -\delta^{\alpha\beta} \hat{E}_\beta{}^i \partial_i \hat{\Omega} & \text{for } \gamma = 1, \\ 0 & \text{for } 1 < \gamma < 2, \end{cases} \quad \hat{Q}^\alpha = \begin{cases} 3\gamma \hat{\Omega} \hat{v}^\alpha & \text{for } 1 \leq \gamma < \frac{4}{3}, \\ 12\hat{\Omega} \hat{v}^\alpha / (3 + \hat{v}_\beta \hat{v}^\beta) & \text{for } \gamma = \frac{4}{3}, \\ 3\hat{\Omega} \hat{v}^\alpha & \text{for } \frac{4}{3} < \gamma < 2. \end{cases} \quad (83)$$

The constraints $(C_\Lambda)_\alpha$ and (C_G) have been used in App. B to give the coefficients in Eqs. (69) and (71), respectively.⁴

3.3 Features of asymptotic to de Sitter G_0 cosmologies

In this section we investigate certain features of asymptotic to de Sitter G_0 cosmologies, as described by the asymptotic expansions (68)–(74). Our first goal is to show that these expansions represent a general class of perfect fluid G_0 cosmologies. We accomplish this by showing that the expansions (68)–(74) contain 8 freely specifiable functions of the spatial coordinates x^i , the same number that appear in the initial data for a general cosmological solution of Einstein's field equations with a perfect fluid matter source (see LK, p. 188).

Essential arbitrary functions

The nine coefficients $\hat{E}_\alpha{}^i$ in the expansions (68)–(74) can be chosen as suitably differentiable arbitrary functions of the x^i , which then successively determine \hat{A}^α , $\hat{N}_{\alpha\beta}$, $\hat{\Omega}_k$ and $\hat{S}_{\alpha\beta}$ via Eqs. (75), (76), (80) and (81). The five shear rate coefficients $\hat{\Sigma}_{\alpha\beta}$ and the coefficient $\hat{\Omega}$ in the density parameter can also be chosen arbitrarily, while the peculiar velocity coefficients \hat{v}^α are determined algebraically by the Codacci constraint (82), employing Eq. (83). As mentioned in Sec. 2, the separable volume gauge (11) leaves a freedom in the choice of the 1-parameter family of spacelike 3-surfaces $\mathcal{S}:\{t = \text{constant}\}$. A coordinate transformation of the form⁵

$$\tilde{t} = t + \varphi(x^i) + \mathcal{O}(e^{-2t}), \quad \tilde{x}^i = x^i - \frac{1}{2} \left(\frac{\Lambda}{3} \right)^{-1} \hat{g}^{ij} \partial_j \varphi e^{-2t} + \mathcal{O}(e^{-3t}), \quad (84)$$

where $\hat{g}^{ij} = \ell_0^{-2} \delta^{\alpha\beta} \hat{E}_\alpha{}^i \hat{E}_\beta{}^j$, preserves the separable volume gauge to leading order. This freedom can be used to fix $\hat{\Omega}$, although we do not choose to do so in general. Three of the $\hat{E}_\alpha{}^i$ can be eliminated by the frame rotations (15), and three by the spatial coordinate transformations (16), leaving three essentially arbitrary functions in the $\hat{E}_\alpha{}^i$. The asymptotic expansions (68)–(74) thus contain 8 essentially arbitrary functions of the spatial coordinates. That is, they represent a general class of perfect fluid G_0 cosmologies that are future asymptotic to the de Sitter solution.

Metric expansion

We now derive an asymptotic expansion for the spacetime metric in order to relate our results to the work of other researchers, in particular that of Starobinskiĭ [30]. We find that the metric expansion sheds light on the nature of large-scale spatial inhomogeneity in G_0 cosmologies that are future asymptotic to the de Sitter solution.

In the separable volume gauge, the line element has the form

$$ds^2 = -H^{-2} dt^2 + g_{ij} dx^i dx^j, \quad (85)$$

where [see Paper I, Eq. (163)]

$$g_{ij} = H^{-2} \delta_{\alpha\beta} E^\alpha{}_i E^\beta{}_j. \quad (86)$$

⁴Note that up to (and including) order e^{-2t} , the present expansion also satisfies the gauge conditions defining the constant mean curvature ($r_\alpha = 0$) and synchronous ($\dot{U}_\alpha = 0$) temporal gauges, respectively.

⁵See Starobinskiĭ [30] for the analogous synchronous gauge-preserving transformation. The function $\varphi(x^i)$ can be interpreted in terms of a boost of the original frame by the boost function $w^\alpha(t, x^i)$ via $w^\alpha = -\delta^{\alpha\beta} (\hat{E}_\beta{}^i \partial_i \varphi) e^{-t} + \mathcal{O}(e^{-2t})$.

Equations (68)–(74) are substituted into Eq. (12) to give an improved expansion for q , and then into Eq. (18) to give an expansion for $\partial_t E_\alpha^i$, which can be integrated. In the case $\gamma < \frac{4}{3}$, we obtain

$$E_\alpha^i = \hat{E}_\beta^i e^{-t} \left[\delta_\alpha^\beta - \frac{1}{2} \left(3\hat{\mathcal{S}}_\alpha^\beta + \hat{\Omega}_k \delta_\alpha^\beta \right) e^{-2t} + \frac{1}{3} \hat{\Sigma}_\alpha^\beta e^{-3t} - \frac{1}{2} \hat{\Omega} \delta_\alpha^\beta e^{-3\gamma t} + \mathcal{O}(e^{-4t}) \right]. \quad (87)$$

The components E_α^i of the corresponding Hubble-normalized 1-forms form the inverse of the matrix E_α^i , and are given by

$$E^\alpha_i = \hat{E}^\beta_i e^t \left[\delta^\alpha_\beta + \frac{1}{2} \left(3\hat{\mathcal{S}}^\alpha_\beta + \hat{\Omega}_k \delta^\alpha_\beta \right) e^{-2t} - \frac{1}{3} \hat{\Sigma}^\alpha_\beta e^{-3t} + \frac{1}{2} \hat{\Omega} \delta^\alpha_\beta e^{-3\gamma t} + \mathcal{O}(e^{-4t}) \right], \quad (88)$$

with \hat{E}^β_i and \hat{E}_β^i related by Eqs. (77). It also follows from Eq. (9) that the Hubble scalar H has the expansion

$$H = \ell_0^{-1} \left[1 + \frac{1}{2} \hat{\Omega}_k e^{-2t} + \frac{1}{2} \hat{\Omega} e^{-3\gamma t} + \mathcal{O}(e^{-4t}) \right]. \quad (89)$$

We then substitute Eqs. (88) and (89) into Eq. (86) to obtain the following expansion for the 3-metric g_{ij} :

$$g_{ij} = \ell_0^2 e^{2t} \left[\hat{g}_{ij} + 3\hat{\mathcal{S}}_{ij} e^{-2t} - \frac{2}{3} \hat{\Sigma}_{ij} e^{-3t} + \mathcal{O}(e^{-4t}) \right], \quad (90)$$

where

$$\hat{g}_{ij} = \delta_{\alpha\beta} \hat{E}^\alpha_i \hat{E}^\beta_j, \quad \hat{\mathcal{S}}_{ij} = \hat{\mathcal{S}}_{\alpha\beta} \hat{E}^\alpha_i \hat{E}^\beta_j, \quad \hat{\Sigma}_{ij} = \hat{\Sigma}_{\alpha\beta} \hat{E}^\alpha_i \hat{E}^\beta_j. \quad (91)$$

Observe that the free functions \hat{E}_α^i determine the coefficients \hat{g}_{ij} in the leading order term in the expansion through Eqs. (77) and (91), and also determine the second term $\hat{\mathcal{S}}_{ij}$, while the free functions $\hat{\Sigma}_{ij}$ enter at a higher order. Note that up to order $\mathcal{O}(e^{-4t})$, the matter coefficient $\hat{\Omega}$ only enters the spacetime metric (85) through H in Eq. (89). It can be shown that the expansions (89) and (90) are also valid for γ satisfying $\frac{4}{3} \leq \gamma < 2$, with the difference that the matter coefficient enters into the $\mathcal{O}(e^{-4t})$ term in Eq. (89).

Our results provide a confirmation of the ansatz for the metric expansion given by Starobinskii [30], Eq. (2). Some of the details differ, however, due to the fact that Starobinskii employs the synchronous gauge. We have established consistency with his results by performing a coordinate transformation of the form

$$\tilde{t} \equiv T = \int H^{-1} dt + \dots, \quad \tilde{x}^i = x^i + \dots. \quad (92)$$

We have also derived Starobinskii's metric expansion directly in the synchronous gauge, using our integration method.

The most striking feature of the expansion (90) is the fact that the leading order coefficient in the expansion is an arbitrary 3-metric, \hat{g}_{ij} , whereas the spacelike 3-surfaces $\mathcal{S}:\{t = \text{constant}\}$ in the de Sitter line element (52) are intrinsically flat. The generality of the leading order coefficient \hat{g}_{ij} in the metric expansion raises an apparent paradox — a G_0 cosmology that is future asymptotic to the spatially homogeneous and isotropic de Sitter solution in the sense of the definition in Subsec. 3.1 can exhibit substantial spatial inhomogeneity. We will show that this paradox is resolved by the existence of event horizons.

Event horizons

In cosmology an event horizon for a fundamental observer can be thought of as constituted by the observer's past light cone at $t = \infty$ (see Hawking and Ellis [14], p. 129). In order to establish the existence of event horizons, we need to determine the asymptotic form of geodesic null congruences. These are governed by Eq. (51), with the K^α determined by Eqs. (48)–(50).

In App. D we show that any null geodesic has the asymptotic form

$$x^i(t) = x_\infty^i - \hat{K}^\alpha \hat{E}_\alpha^i(x_\infty^j) e^{-t} + \mathcal{O}(e^{-2t}), \quad (93)$$

for constants x_∞^i and \hat{K}^α . For given x_∞^i , the family of null geodesics then form an event horizon for the fundamental observer whose worldline is $x^i = x_\infty^i$. The spatial distance from the observer to her/his event horizon is given by

$$d_H(t) = \int_0^1 \sqrt{g_{ij} \frac{dy^i}{ds} \frac{dy^j}{ds}} ds, \quad (94)$$

where $y^i = y^i(s)$ (with $0 \leq s \leq 1$) describes a spacelike geodesic from x_∞^i to $x^i(t)$, for fixed t . In the asymptotic regime, the spacelike geodesic is basically a straight line, and is approximated by the null geodesic (93) projected onto a spacelike 3-surface $\mathcal{S}:\{t = \text{constant}\}$. It follows from Eq. (93) that

$$d_H(t) = \ell_0 + \mathcal{O}(e^{-t}), \quad (95)$$

where ℓ_0 is given in Eq. (54).

One can always introduce local coordinates at $x^i = x_\infty^i$ such that (see e.g., Schutz [29], p. 156)

$$\hat{g}_{ij}(x^m) = \ell_0^2 \delta_{ij} + \frac{\partial^2 \hat{g}_{ij}(x_\infty^m)}{\partial x^k \partial x^l} (x^k - x_\infty^k) (x^l - x_\infty^l) + \dots \quad (96)$$

For points within the event horizon the approximation $\hat{g}_{ij}(x^m) \approx \ell_0^2 \delta_{ij}$ becomes increasingly accurate as $t \rightarrow \infty$, due to Eq. (93). In other words, *within* the event horizon of a particular fundamental observer, the spacetime metric asymptotically approaches the de Sitter metric.⁶ Since \hat{g}_{ij} is general, however, the above approximation cannot be done simultaneously at all points, reflecting the spatial inhomogeneity of the G_0 cosmology at super-horizon scales. We emphasize that the spatial inhomogeneity does not diminish as $t \rightarrow \infty$ — it is the observer who sees successively smaller portions of that spatial inhomogeneity.

Radiation bifurcation

A noteworthy feature of the asymptotic expansion (68)–(74) is that for the physically important case of incoherent radiation ($\gamma = \frac{4}{3}$), a bifurcation occurs that affects the peculiar velocity \mathbf{v} of the perfect fluid relative to the spacelike 3-surfaces $\mathcal{S}:\{t = \text{constant}\}$.

If $1 \leq \gamma < \frac{4}{3}$, the components v^α tend to zero, and as a result the fluid 4-velocity vector field $\tilde{\mathbf{u}}$ asymptotically coincides with the vorticity-free timelike reference congruence \mathbf{e}_0 . For incoherent radiation, the v^α do not tend to zero in general, and v^2 can approach any value between 0 and 1. If $\frac{4}{3} < \gamma < 2$, v^2 tends to the value 1 as $t \rightarrow \infty$, which means that the speed of the fluid relative to the fundamental observers comoving with \mathbf{e}_0 approaches the speed of light.

It is of considerable interest to investigate the limit behavior of the fluid kinematical variables, as perceived by the fundamental observers. Using the boost transformation laws provided in App. F, we find that the future asymptotic limits for the Hubble-normalized fluid kinematical scalars (defined by Eqs. (270)–(272) in App. F) are

$$\lim_{t \rightarrow \infty} (\dot{\tilde{U}}^2, \tilde{\Sigma}^2, \tilde{W}^2) = (0, 0, 0) \quad \text{for } 1 \leq \gamma < \frac{4}{3}. \quad (97)$$

$$\lim_{t \rightarrow \infty} (\dot{\tilde{U}}^2, \tilde{\Sigma}^2, \tilde{W}^2) = \left[\frac{1}{3} \tilde{v}^2, 0, 0 \right] \quad \text{for } \gamma = \frac{4}{3}. \quad (98)$$

$$\lim_{t \rightarrow \infty} (\dot{\tilde{U}}^2, \tilde{\Sigma}^2, \tilde{W}^2) = \left[3(\gamma - 1)^2, \frac{1}{4}(3\gamma - 4)^2, 0 \right] \quad \text{for } \frac{4}{3} < \gamma < 2. \quad (99)$$

This extends previous results given by Goliath and Ellis [11] for some special SH cases with cosmological constant to the general G_0 case (see also Goliath and Nilsson [12] and Raychaudhuri and Modak [24]). The context of spatial inhomogeneity stresses the isotropization issue. We have to ask ourselves isotropization along which timelike congruence? In the present context there is no isotropization along the timelike reference congruence \mathbf{e}_0 when $\gamma > \frac{4}{3}$, but this does not say anything about whether the models isotropize along the fluid 4-velocity vector field in this case, which, perhaps, is the physically best motivated congruence to consider. However, to determine whether this is the case or not takes us outside the scope of the present paper, and so we leave this issue for future studies. We note that when $\gamma < \frac{4}{3}$, the fluid 4-velocity vector field asymptotically coincides with the timelike reference congruence and, hence, one does have isotropization in this case.

4 Flat FL-like past asymptotics

In this section we give the detailed asymptotic form of the Hubble-normalized variables for G_0 cosmologies that approach the flat FL solution at early times.

⁶Boucher and Gibbons [4] have clarified this matter in a different way, by transforming the de Sitter metric to a static form, valid within the event horizon of a particular fundamental observer.

4.1 G_0 cosmologies past asymptotic to the flat FL solution

The flat FL solution of Einstein's field equations describes a cosmological model with perfect fluid matter source, whose density parameter has the constant value $\Omega = 1$. The line element and the fluid 4-velocity vector field are given by

$$ds^2 = -dT^2 + \ell_0^2 \left(\frac{T}{T_0} \right)^{4/(3\gamma)} (dx^2 + dy^2 + dz^2) , \quad (100)$$

$$\hat{\mathbf{u}} = \partial_T , \quad (101)$$

employing clock time T as one of the local coordinates. The matter energy density and pressure are

$$\mu = \frac{4}{3\gamma^2} T^{-2} , \quad p = (\gamma - 1) \mu . \quad (102)$$

The Hubble scalar is

$$H = \frac{2}{3\gamma} T^{-1} , \quad (103)$$

while the deceleration parameter is given by

$$q = \frac{1}{2}(3\gamma - 2) ; \quad (104)$$

the latter is thus constant and positive. We find it convenient to use a conformal time coordinate η , defined by

$$\eta := \eta_0 \left(\frac{T}{T_0} \right)^{(3\gamma-2)/3\gamma} , \quad \eta_0 := \frac{3\gamma}{(3\gamma-2)} \frac{T_0}{\ell_0} , \quad (105)$$

with the unit of [length] ℓ_0 and the constant T_0 chosen to satisfy $\eta_0 = 1$, i.e.,

$$\ell_0 = \frac{3\gamma}{(3\gamma-2)} T_0 . \quad (106)$$

The line element is then cast into the form

$$ds^2 = \ell_0^2 \eta^{4/(3\gamma-2)} (-d\eta^2 + dx^2 + dy^2 + dz^2) , \quad (107)$$

and the Hubble scalar is

$$H = \frac{2}{3\gamma} T_0^{-1} \eta^{-3\gamma/(3\gamma-2)} . \quad (108)$$

We introduce the dynamical time coordinate t via

$$\left(\frac{T}{T_0} \right)^{2/(3\gamma)} = e^t , \quad (109)$$

or, equivalently,

$$\eta = e^{\frac{1}{2}(3\gamma-2)t} . \quad (110)$$

Relative to the natural orthonormal frame associated with the line element (100), the Hubble-normalized variables are [using Eq. (106)]

$$E_\alpha{}^i = \frac{1}{2} (3\gamma - 2) e^{\frac{1}{2}(3\gamma-2)t} \delta_\alpha{}^i \quad (111)$$

$$\Sigma_{\alpha\beta} = 0 , \quad A^\alpha = 0 , \quad N_{\alpha\beta} = 0 , \quad (112)$$

$$\Omega = 1 , \quad \Omega_\Lambda = 0 . \quad (113)$$

We note that the volume density \mathcal{V} is given by

$$\mathcal{V} = \ell_0^3 e^{3t} , \quad (114)$$

so that the separable volume gauge conditions are satisfied.

Observe that the frame variables satisfy $\lim_{t \rightarrow -\infty} E_\alpha^i = 0$. Thus, the flat FL solution is described by an orbit in the Hubble-normalized state space that is past asymptotic to an equilibrium point on the silent boundary, $E_\alpha^i = 0$.

Motivated by Eqs. (111)–(113), we say that a G_0 cosmology is past asymptotic to the flat FL solution if the following limits are satisfied:

$$\lim_{t \rightarrow -\infty} (E_\alpha^i, r_\alpha)^T = \mathbf{0} \quad (115)$$

$$\lim_{t \rightarrow -\infty} (\Sigma_{\alpha\beta}, A^\alpha, N_{\alpha\beta})^T = \mathbf{0} \quad (116)$$

$$\lim_{t \rightarrow -\infty} \Omega = 1, \quad \lim_{t \rightarrow -\infty} \Omega_\Lambda = 0. \quad (117)$$

We note that for the flat FL solution the peculiar velocity variables v^α are zero. We will show that if the limits (115)–(117) are satisfied, together with certain technical restrictions, then the constraints imply that the v^α tend to zero as $t \rightarrow -\infty$. We now consider the class of G_0 cosmologies which are past asymptotic to the flat FL solution, and present the asymptotic form of the Hubble-normalized variables as $t \rightarrow -\infty$.

4.2 Asymptotic expansions

In App. C, we derive the asymptotic form of the Hubble-normalized variables for G_0 cosmologies that are past asymptotic to the flat FL solution, in the sense of Eqs. (115)–(117). It is necessary, as in Subsec. 3.2, to impose certain restrictions on the spatial derivatives; these are given in App. C.

We now give the asymptotic expansions. For brevity we use η instead of $e^{\frac{1}{2}(3\gamma-2)t}$ [see Eq. (110)].

$$(E_\alpha^i, A^\alpha, N^{\alpha\beta})^T = (\hat{E}_\alpha^i, \hat{A}^\alpha, \hat{N}^{\alpha\beta})^T \eta + \mathcal{O}(\eta^3) \quad (118)$$

$$r_\alpha = -\frac{1}{2} (\hat{E}_\alpha^i \partial_i \hat{\Omega}_k) \eta^3 + \mathcal{O}(\eta^5 + \eta^{3+\frac{4}{(3\gamma-2)}}) \quad (119)$$

$$\Sigma^{\alpha\beta} = -\frac{6}{(3\gamma+2)} \hat{S}^{\alpha\beta} \eta^2 + \mathcal{O}(\eta^4) \quad (120)$$

$$\Omega_\Lambda = \frac{\Lambda}{3\hat{H}^2} \eta^{2+\frac{4}{(3\gamma-2)}} \left[1 - \hat{\Omega}_k \eta^2 + \mathcal{O}(\eta^4 + \eta^{2+\frac{4}{(3\gamma-2)}}) \right] \quad (121)$$

$$\Omega = 1 - \hat{\Omega}_k \eta^2 + \mathcal{O}(\eta^4 + \eta^{2+\frac{4}{(3\gamma-2)}}) \quad (122)$$

$$v^\alpha = \frac{1}{(3\gamma+2)} \delta^{\alpha\beta} (\hat{E}_\beta^i \partial_i \hat{\Omega}_k) \eta^3 + \mathcal{O}(\eta^5 + \eta^{3+\frac{4}{(3\gamma-2)}}), \quad (123)$$

where \hat{H} is the leading order coefficient in the asymptotic expansion of H :

$$H = \hat{H} \eta^{-[1+\frac{2}{(3\gamma-2)}]} \left[1 + \frac{1}{2} \hat{\Omega}_k \eta^2 + \mathcal{O}(\eta^4 + \eta^{2+\frac{4}{(3\gamma-2)}}) \right]. \quad (124)$$

In the derivation in App. C, we used the freedom to re-define the 1-parameter family of spacelike 3-surfaces $\mathcal{S}: \{t = \text{constant}\}$ while preserving the separable volume gauge to set

$$\hat{H} = \frac{2}{3\gamma} T_0^{-1}, \quad (125)$$

so that $\hat{H} = \text{constant} > 0$. In addition, Eq. (10) and the constraints (\mathcal{C}_G) and $(\mathcal{C}_C)^\alpha$ were used in App. C to give the coefficients in Eqs. (119), (122) and (123), respectively.⁷

As in Subsec. 3.2 before, the coefficients \hat{A}^α , $\hat{N}_{\alpha\beta}$, $\hat{\Omega}_k$ and $\hat{S}_{\alpha\beta}$ are determined by the \hat{E}_α^i according to Eqs. (75), (76), (80) and (81). Likewise, the 3-Ricci curvature variables have the asymptotic expansions

$$\Omega_k = \eta^2 \left[\hat{\Omega}_k + \mathcal{O}(\eta^2) \right] \quad (126)$$

$$S_{\alpha\beta} = \eta^2 \left[\hat{S}_{\alpha\beta} + \mathcal{O}(\eta^2) \right]. \quad (127)$$

⁷Note that up to (and including) order η^2 , the present expansion also satisfies the gauge conditions defining the constant mean curvature ($r_\alpha = 0$), synchronous ($\dot{U}_\alpha = 0$) and fluid-comoving ($v^\alpha = 0$) temporal gauges, respectively.

4.3 Features of asymptotic to flat FL G_0 cosmologies

In this section we investigate certain features of the class of G_0 cosmologies that are past asymptotic to the flat FL solution. One of our goals is to show that this class of cosmologies admit an isotropic initial singularity in the sense of Goode and Wainwright [13] (see also Newman [20] and Anguige and Tod [1]).

Essential arbitrary functions

Of the position-dependent coefficients in Eqs. (118)–(123), only the nine $\hat{E}_\alpha{}^i$ can be chosen arbitrarily, as suitably differentiable functions of the x^i . However, three of them can be eliminated by the frame rotations (15), and three by the spatial coordinate transformations (16). Thus, the asymptotic expansion contains 3 essentially arbitrary functions of the spatial coordinates, as compared with 8 for a general class of perfect fluid G_0 cosmologies. These models thus form a set of measure zero in the Hubble-normalized state space.

Metric expansion

We now derive an asymptotic expansion for the spacetime metric in order to relate our results to the work of other researchers, in particular that of LK and Goode and Wainwright [13]. We find that the metric expansion sheds light on the nature of large-scale spatial inhomogeneity in G_0 cosmologies that are past asymptotic to the flat FL solution.

In the separable volume gauge, the line element has the form given in Eq. (85). Equation (18) gives an expansion for $\partial_t E_\alpha{}^i$, which can be integrated to give

$$E_\alpha{}^i = \hat{E}_\beta{}^i \eta \left[\delta_\alpha{}^\beta + \left(\frac{6}{(3\gamma-2)(3\gamma+2)} \hat{S}_\alpha{}^\beta - \frac{1}{2} \hat{\Omega}_k \delta_\alpha{}^\beta \right) \eta^2 + \mathcal{O}(\eta^4 + \eta^{2+\frac{4}{(3\gamma-2)}}) \right]. \quad (128)$$

Following the steps taken in Subsec. 3.3, we obtain the metric expansion

$$g_{ij} = \eta^{4/(3\gamma-2)} \left[\hat{g}_{ij} - \frac{12}{(3\gamma-2)(3\gamma+2)} \hat{S}_{ij} \eta^2 + \mathcal{O}(\eta^4 + \eta^{2+\frac{4}{(3\gamma-2)}}) \right], \quad (129)$$

where

$$\hat{g}_{ij} = \hat{H}^{-2} \delta_{\alpha\beta} \hat{E}^\alpha{}_i \hat{E}^\beta{}_j, \quad \hat{S}_{ij} = \hat{H}^{-2} \hat{S}_{\alpha\beta} \hat{E}^\alpha{}_i \hat{E}^\beta{}_j. \quad (130)$$

Our result provides a rigorous confirmation of the ansatz for the metric expansion made by LK, Eq. (4.1), for the case of incoherent radiation. Some of the details differ, however, due to the fact that LK employ the synchronous gauge. We have established consistency with their results by performing a coordinate transformation of the form given in Eqs. (92). We have also derived LK's metric expansion directly in the synchronous gauge, using our integration method.

Isotropic initial singularities

We briefly digress to introduce the notion of an isotropic initial singularity; see Ref. [13]. A cosmological model⁸ is said to admit an *isotropic initial singularity* if the physical spacetime metric \mathbf{g} is conformal to an unphysical metric $\tilde{\mathbf{g}}$:

$$\mathbf{g} = \chi^2 \tilde{\mathbf{g}}, \quad (131)$$

where $\chi = \chi(\eta)$ is a differentiable function of a time coordinate η and satisfies $\chi(0) = 0$, while the conformal metric $\tilde{\mathbf{g}}$ is regular on the spacelike 3-surface $\tilde{\mathcal{S}}: \{\eta = 0\}$. The conformal factor $\chi(\eta)$ is also required to satisfy

$$\lim_{\eta \rightarrow 0^+} \frac{\chi'(\eta)}{\chi(\eta)} = \infty \quad (132)$$

$$\lim_{\eta \rightarrow 0^+} \frac{\chi''(\eta)}{\chi(\eta)} \left(\frac{\chi'(\eta)}{\chi(\eta)} \right)^{-2} = -\frac{1}{2} (3\gamma - 4). \quad (133)$$

⁸We assume that the matter source is a perfect fluid satisfying Eqs. (1) and (2).

It follows from Eqs. (85), (124) and (129) that the spacetime metric for the class of G_0 cosmologies that are past asymptotic to the flat FL solution does satisfy Eqs. (132) and (133) with

$$\chi = \ell_0 \eta^{2/(3\gamma-2)} , \quad (134)$$

where η is the conformal time coordinate.

Particle horizons

The generality of the leading order coefficient \hat{g}_{ij} in the metric expansion (129), which should be compared with Eq. (90), raises an apparent paradox — a G_0 cosmology that is past asymptotic to the spatially homogeneous and isotropic flat FL solution in the sense of the definition in Subsec. 4.1 can exhibit substantial spatial inhomogeneity. We will show that this paradox is resolved by the existence of particle horizons.

In order to establish the existence of particle horizons, we need to determine the asymptotic form of geodesic null congruences. These are governed by Eq. (51), with the K^α determined by Eqs. (48)–(50).

In App. E we show that any past-oriented null geodesic has the following asymptotic form as $\eta \rightarrow 0$:

$$x^i(\eta) = x_{\text{BB}}^i + \frac{2}{(3\gamma-2)} \hat{K}^\alpha \hat{E}_\alpha{}^i(x_{\text{BB}}^j) \eta + \mathcal{O}(\eta^2) , \quad (135)$$

for constants x_{BB}^i and \hat{K}^α . The constants x_{BB}^i give the point of termination of the null geodesic at the singularity $\eta = 0$. Thus the past null cone at time η for the fundamental observer whose worldline is $x^i = x_0^i$ will intersect the singularity $\eta = 0$, thereby defining a particle horizon, which is formed by the fundamental worldlines given by $x^i = x_{\text{BB}}^i$. The spatial distance from the observer to her/his particle horizon at time η is given by

$$d_{\text{H}}(\eta) = \int_0^1 \sqrt{g_{ij} \frac{dy^i}{ds} \frac{dy^j}{ds}} ds , \quad (136)$$

where $y^i = y^i(s)$ (with $0 \leq s \leq 1$) describes a spacelike geodesic from x_{BB}^i to $x^i(\eta)$, for fixed η . As in the derivation of Eq. (95), it follows from Eq. (135) that

$$d_{\text{H}}(\eta) = \eta^{1+\frac{2}{(3\gamma-2)}} [\ell_0 + \mathcal{O}(\eta)] , \quad (137)$$

where ℓ_0 is given in Eq. (106).

One can introduce local coordinates at $x^i = x_{\text{BB}}^i$ such that

$$\hat{g}_{ij}(x^m) = \ell_0^2 \delta_{ij} + \frac{\partial^2 \hat{g}_{ij}(x_{\text{BB}}^m)}{\partial x^k \partial x^l} (x^k - x_{\text{BB}}^k) (x^l - x_{\text{BB}}^l) + \dots . \quad (138)$$

For points within the particle horizon the approximation $\hat{g}_{ij}(x^m) \approx \ell_0^2 \delta_{ij}$ becomes increasingly accurate as $\eta \rightarrow 0$, due to Eq. (135). In other words, *within* the particle horizon of a particular fundamental observer, the spacetime metric is close to the flat FL metric. Since \hat{g}_{ij} is general, however, the above approximation cannot be done simultaneously at all points, reflecting the spatial inhomogeneity of the G_0 cosmology at super-horizon scales. We emphasize that the spatial inhomogeneity does not diminish — it is the observer who sees successively smaller portions of that spatial inhomogeneity.

5 Concluding remarks

In this paper, we have given a unified analysis, within the framework of the Hubble-normalized state space, of G_0 cosmologies that undergo asymptotic isotropization, either at late times (asymptotic to the de Sitter solution, with $\ell \rightarrow \infty$) or near the initial singularity (asymptotic to the flat FL solution, with $\ell \rightarrow 0$).

The analysis reveals a number of common features as regards the asymptotic behavior in the two classes, as well as a number of significant differences, which emerge most clearly if we write the asymptotic expansions in Subsec. 3.2 in terms of the conformal time coordinate of the de Sitter line element, given by Eq. (57). We note that the length scale factor is given by

$$\ell(\eta) = \ell_0 \eta^{-1} , \quad (139)$$

for the de Sitter solution, and by

$$\ell(\eta) = \ell_0 \eta^{2/(3\gamma-2)} , \quad (140)$$

for the flat FL solution. Thus, for both asymptotes $\eta \rightarrow 0^+$ in the asymptotic regimes under consideration.

Firstly, the shear rate $\Sigma_{\alpha\beta}$ and anisotropic 3-Ricci curvature $\mathcal{S}_{\alpha\beta}$ have the same leading asymptotic dependence on η as $\eta \rightarrow 0^+$:

$$\Sigma_{\alpha\beta} = C \hat{\mathcal{S}}_{\alpha\beta} \eta^2 + \mathcal{O}(\eta^{3+\delta}) \quad (141)$$

$$\mathcal{S}_{\alpha\beta} = \hat{\mathcal{S}}_{\alpha\beta} \eta^2 + \mathcal{O}(\eta^{3+\delta}) , \quad (142)$$

where $\delta \geq 0$, and $C = -3$ in the de Sitter case while $C = -6/(3\gamma + 2)$ in the flat FL case.

Secondly, there is a common asymptotic form for the line element, namely

$$ds^2 = \ell^2(\eta) \left[-d\eta^2 + \ell_0^{-2} \hat{g}_{ij} dx^i dx^j + \mathcal{O}(\eta^2) \right] , \quad (143)$$

as $\eta \rightarrow 0^+$, where $\ell(\eta)$ is the length scale factor for the de Sitter solution or for the flat FL solution. Likewise, the Hubble scalar has the common form

$$H = H_{\text{asympt}}(\eta) \left[1 + \frac{1}{2} \hat{\Omega}_k \eta^2 + \mathcal{O}(\eta^{3+\delta}) \right] , \quad (144)$$

where $\delta \geq 0$, and $H_{\text{asympt}}(\eta)$ is the Hubble scalar for the de Sitter solution or for the flat FL solution, respectively [see Eqs. (55), (89), (108), (124) and (125)]. Here $\hat{\Omega}_k$ is the leading order coefficient in the expansion for the 3-Ricci curvature scalar Ω_k , which has a common form for the two classes, namely,

$$\Omega_k = \hat{\Omega}_k \eta^2 + \mathcal{O}(\eta^4) . \quad (145)$$

The key feature of the asymptotic line element (143) is the presence of the arbitrary 3-metric $\hat{g}_{ij}(x^k)$. We have shown that the spatial inhomogeneity that it generates is significant only at super-horizon scales (the event horizon for G_0 cosmologies that are future asymptotic to the de Sitter solution and the particle horizon for G_0 cosmologies that are past asymptotic to the flat FL solution).

The common asymptotic features of the two classes can be understood to some extent within the context of the Hubble-normalized state space. For each class, the asymptotic state is described by an equilibrium point on the silent boundary,

$$E_\alpha{}^i = 0 ,$$

[see Paper I, Eq. (75)], which is associated with an isotropic and conformally flat solution of Einstein's field equations. Use of a conformal time coordinate η , which tends to zero on the silent boundary for both classes, leads to common asymptotic decay rates for its various Hubble-normalized quantities. The asymptotically silent dynamics ($E_\alpha{}^i \rightarrow 0$) is also responsible for the existence of an event horizon and a particle horizon, as shown in Subsecs. 3.2 and 4.2.

The differences between the two classes originate in the asymptotic behavior of the deceleration parameter q , which is negative in the de Sitter case ($q \rightarrow -1$) and positive in the flat FL case ($q \rightarrow \frac{1}{2}(3\gamma - 2)$). This difference affects the higher-order terms in the asymptotic expansion of the shear rate $\Sigma_{\alpha\beta}$. In the de Sitter case, there is a term $\hat{\Sigma}_{\alpha\beta} \eta^3$, where the $\hat{\Sigma}_{\alpha\beta}$ are freely specifiable functions, while in the flat FL case, this term does not appear.⁹ The absence of the $\hat{\Sigma}_{\alpha\beta}$ -term, of course, accounts for the difference in the number of freely specifiable functions in the two cases. This difference is also reflected in the asymptotic expansions for the Hubble-normalized electric and magnetic parts of the Weyl curvature [see Paper I, Eqs. (152) and (153)]. For the electric Weyl curvature we have in the de Sitter case

$$\mathcal{E}_{\alpha\beta} = \frac{1}{3} \hat{\Sigma}_{\alpha\beta} \eta^3 + \mathcal{O}(\eta^4) , \quad (146)$$

and in the flat FL case,

$$\mathcal{E}_{\alpha\beta} = \frac{3\gamma}{(3\gamma + 2)} \hat{\mathcal{S}}_{\alpha\beta} \eta^2 + \mathcal{O}(\eta^4) . \quad (147)$$

For the magnetic Weyl curvature we have in both cases

$$\mathcal{H}_{\alpha\beta} = -\hat{\mathcal{C}}_{\alpha\beta} \eta^3 + \mathcal{O}(\eta^4) , \quad (148)$$

⁹The second shear rate mode in this case is a growing mode (into the past), and the assumption of the isotropization requires the corresponding coefficient to be zero.

where $\hat{\mathcal{C}}_{\alpha\beta}$ is given by the “hatted” version of Eq. (33), with r_α set to zero.

The difference between the de Sitter and the flat FL cases as regards the number of freely specifiable functions can be interpreted in a dynamical systems context. We have already noted that both the de Sitter solution (52) and the flat FL solution (100) determine equilibrium points on the silent boundary in the Hubble-normalized state space. It follows that the orbits of G_0 cosmologies that are future asymptotic to the de Sitter solution (respectively, past asymptotic to the flat FL solution) are asymptotic to these equilibrium points. The asymptotic expansion contains 8 arbitrary functions (the maximum possible number) in the de Sitter case, which confirms that the de Sitter equilibrium point is asymptotically stable, as proved in App. B (see Stage 0). On the other hand, the presence of only 3 arbitrary functions in the flat FL case implies that the flat FL equilibrium point has both a stable and an unstable manifold (into the past), and our asymptotic solutions describe the stable manifold.

A major accomplishment of the present paper is to provide derivations of the detailed asymptotic properties of G_0 cosmologies that undergo asymptotic isotropization, instead of simply making *ad hoc* assumptions about the form of the metric, as has been done previously. Our derivations do, however, depend in a significant way on the assumption that the spatial partial derivatives of the Hubble-normalized variables remain bounded in the asymptotic regimes. It is this assumption that enables us to rely exclusively on analytic methods for systems of ODE.

It would certainly be desirable to attempt to use methods from the theory of PDE to prove the validity of the assumption of bounded spatial derivatives for cosmological models that undergo asymptotic isotropization. Recent experience with G_2 cosmologies, which we now describe, gives some cause for optimism. Firstly, it is known that in G_2 cosmologies spatial partial derivatives can diverge on approach to the initial singularity, through the creation of so-called Gowdy spikes (see Paper I, Subsec. 4.1, and Ref. [25]). This spatial structure is created, however, by the local instability of the Kasner solutions. Since these solutions do not play a rôle in the present context (the approach to an isotropic initial singularity), we do not expect spatial structure of this nature to develop. Secondly, at late times, support for our assumption is provided by numerical simulations¹⁰ of G_2 cosmologies, which do not show the development of large spatial derivatives.

A second limitation of our analysis is that it is local in nature in that we restrict our considerations to some open subset of the integral curves of the timelike reference congruence \mathbf{e}_0 . This restriction is inevitable, however, since in a G_0 cosmology all timelines do not necessarily share the same asymptotic evolution. For example, a G_0 cosmology with positive cosmological constant may approach the de Sitter solution along some timelines but may undergo collapse to a future singularity along others.

Another matter that requires comment is the choice of temporal gauge. In the present paper we have chosen to work with the *separable volume gauge*, defined by Eqs. (11), because we have found it to be a convenient computational gauge. On the other hand, we have been able to transform our asymptotic expansions to the synchronous gauge, and find that the leading order terms, e.g., Eqs. (141)–(148), are unchanged. This calculation thus provides evidence for the *gauge robustness* of our results.

In conclusion, it would be of interest to relate our results to perturbation analyses of the de Sitter solution (see for example Barrow [3] and Bruni *et al* [5]) and of the flat FL solution. We have not been able to clarify the relationship to our satisfaction, particularly in the case of the de Sitter solution, and so we leave this matter for future investigation. In addition, other possible applications of the Hubble-normalized state space framework for G_0 cosmologies presented here that suggest themselves naturally are establishing firm links with mainstream issues in observational cosmology such as the cosmic background radiation, peculiar motions of galaxies, or gravitational lensing.

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¹⁰Woei Chet Lim, unpublished.

A Asymptotic results for ordinary differential equations

Proposition 1

Given

$$\partial_t M_{\alpha\beta} = A_\alpha{}^\gamma M_{\gamma\beta} + B_{\alpha\beta} , \quad (149)$$

where $M_{\alpha\beta}$, $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are Cartesian tensors.¹¹ If

$$\lim_{t \rightarrow \infty} A_{\alpha\beta}(t) = -k\delta_{\alpha\beta} , \quad k > 0 , \quad (150)$$

and

$$B_{\alpha\beta} = \mathcal{O}(e^{-lt}) , \quad l > 0 , \quad (151)$$

then for any given $\epsilon > 0$,

$$M_{\alpha\beta} = \mathcal{O}(e^{(-m+\epsilon)t}) , \quad (152)$$

where $m = \min(k, l)$.

Proof: Introduce

$$M = (M_{\alpha\beta} M^{\alpha\beta})^{1/2} , \quad M_{\alpha\beta} = M S_{\alpha\beta} , \quad \text{where } (S_{\alpha\beta} S^{\alpha\beta})^{1/2} = 1 . \quad (153)$$

It follows from Eqs. (149) that

$$\partial_t M = (S^{\alpha\beta} A_\alpha{}^\gamma S_{\gamma\beta}) M + S^{\alpha\beta} B_{\alpha\beta} . \quad (154)$$

Now Eq. (150) implies that for all $\epsilon > 0$ there exists a t_0 such that

$$|S^{\alpha\beta} (A_\alpha{}^\gamma + k\delta_\alpha{}^\gamma S_{\gamma\beta})| \leq \epsilon , \quad (155)$$

for all $t \geq t_0$, and Eq. (151) implies there exists a $C > 0$ such that

$$|S^{\alpha\beta} B_{\alpha\beta}| \leq C e^{-lt} . \quad (156)$$

It then follows that

$$\partial_t M \leq (-k + \epsilon) M + C e^{-lt} . \quad (157)$$

Multiplying by $e^{(k-\epsilon)t}$ and integrating from $t = t_0$ to t , yields

$$M \leq \left[M_0 + \frac{C e^{-lt_0}}{(-k + \epsilon + l)} \right] e^{(-k+\epsilon)(t-t_0)} + \frac{C e^{-lt_0}}{(k - \epsilon - l)} e^{-l(t-t_0)} , \quad (158)$$

from which the result follows. \square

Proposition 2

Given

$$\partial_t M_{\alpha\beta} = A_\alpha{}^\gamma M_{\gamma\beta} + B_{\alpha\beta} , \quad (159)$$

where $M_{\alpha\beta}$, $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are Cartesian tensors. If

$$\lim_{t \rightarrow \infty} A_{\alpha\beta}(t) = k\delta_{\alpha\beta} , \quad k > 0 , \quad (160)$$

$$B_{\alpha\beta} = \mathcal{O}(e^{-lt}) , \quad l > 0 , \quad (161)$$

and $M_{\alpha\beta}$ is bounded, then

$$M_{\alpha\beta} = \mathcal{O}(e^{-lt}) . \quad (162)$$

¹¹We will also need a vector version of this result, i.e., the DE is of the form

$$\partial_t M^\alpha = A^\alpha{}_\beta M^\beta + B^\alpha .$$

Proof: As in the proof of Proposition 1, we obtain Eqs. (154) and Eq. (155) with k replaced by $-k$, and Eq. (156). It follows from these equations that

$$\partial_t M \geq M(k - \epsilon) - C e^{-lt} . \quad (163)$$

Multiply by $e^{-(k-\epsilon)t}$ and integrate from t to ∞ , knowing that M is bounded, to obtain

$$M \leq \frac{C}{(k - \epsilon + l)} e^{-lt} , \quad (164)$$

and the result follows. \square

Proposition 3

Given

$$\partial_t \mathbf{M} = -k\mathbf{M} + \mathbf{B} , \quad (165)$$

where \mathbf{M} and \mathbf{B} are $m \times n$ matrices and $k > 0$ is constant. If $\mathbf{B} = \mathcal{O}(e^{-lt})$, $l > k$, then

$$\mathbf{M} = \hat{\mathbf{M}} e^{-kt} + \mathcal{O}(e^{-lt}) . \quad (166)$$

Proof: Rewrite the differential equation (DE) as $\partial_t(e^{kt}\mathbf{M}) = e^{kt}\mathbf{B}$, and integrate. \square

B Derivation of asymptotically de Sitter future dynamics

In this appendix we derive the asymptotic expansions (68)–(74) for G_0 cosmologies that are future asymptotic to the de Sitter solution in the sense of Eqs. (65)–(67). For ease of reference, we list the complete set of restrictions below.

A1: E_α^i , A^α , $N_{\alpha\beta}$, r_α , $\Sigma_{\alpha\beta}$, Ω and $\Omega_\Lambda - 1$ are sufficiently small at some initial time which we can take to be $t = 0$.

A2: The partial derivatives of any dimensionless variable V are bounded as $t \rightarrow \infty$. In addition, if V tends to zero, then $\partial_i V = \mathcal{O}(\|V\|)$ as $t \rightarrow \infty$, for fixed x^i , where $\|V\|$ is the magnitude of V .¹²

A3: If $V = \hat{V} e^{-nt} + \mathcal{O}(e^{-mt})$ as $t \rightarrow \infty$, where $m > n > 0$, then $\partial_i V = \partial_i \hat{V} e^{-nt} + \mathcal{O}(e^{-mt})$ as $t \rightarrow \infty$.

Stage 0

We first establish asymptotic stability of the de Sitter solution. The evolution equations for the variables in A1 are of the form

$$\partial_t \mathbf{X} = \mathbf{A}\mathbf{X} + \mathbf{f}(t, \mathbf{X}) , \quad (167)$$

where \mathbf{X} is the vector containing the variables in A1, \mathbf{A} is a constant diagonal real-valued matrix with negative eigenvalues, while $\mathbf{f}(t, \mathbf{X})$ contains products of components in \mathbf{X} and $\partial_\alpha \mathbf{X}$. Assumptions A1 and A2 imply that $\mathbf{f}(t, \mathbf{X})$ is continuous for all $t \geq 0$ and $\mathbf{f}(t, \mathbf{X}) = o(\|\mathbf{X}\|)$ as $\|\mathbf{X}\| \rightarrow 0$. Then Theorem 1.1 in Coddington and Levinson [7], p. 314, implies that $\mathbf{X} = \mathbf{0}$ is asymptotically stable. Thus, we obtain the property

P1: E_α^i , A^α , $N_{\alpha\beta}$, r_α , $\Sigma_{\alpha\beta}$, Ω , $\Omega_\Lambda - 1 \rightarrow 0$ as $t \rightarrow \infty$, for fixed x^i .

We organize the derivation of asymptotic expansions into two stages, in which we repeatedly use the evolution equations and the constraints (\mathcal{C}_G) and $(\mathcal{C}_\Lambda)_\alpha$. In Stage 1, we obtain order estimates for all variables, primarily using Propositions 1 and 2 in App. A. In Stage 2, we use the results from Stage 1 in conjunction with Proposition 3 to obtain the explicit asymptotic expansions. The assumptions A2 and A3 are needed throughout in order to bound the spatial derivative terms in the evolution equations and in the constraints (\mathcal{C}_G) and $(\mathcal{C}_\Lambda)_\alpha$.

¹²For example, $\|E_\alpha^i\| = \sqrt{\delta^{\alpha\beta} E_\alpha^i E_\beta^i}$, for each i , and $\|\Sigma_{\alpha\beta}\| = \Sigma$.

Stage 1

The deceleration parameter q , given by Eq. (12), plays a dominant rôle in the evolution equations. Its limiting value as $t \rightarrow \infty$ follows from P1 and A2:

$$\lim_{t \rightarrow \infty} q = -1 . \quad (168)$$

We begin by considering E_α^i . Using P1 and Eq. (168), the evolution equation (18) for E_α^i assumes the form of Eqs. (149) and (150) with $k = 1$ and $B_{\alpha\beta} = 0$. It follows from Proposition 1 that for any given $\epsilon_E > 0$,

$$E_\alpha^i = \mathcal{O}(e^{(-1+\epsilon_E)t}) , \quad (169)$$

as $t \rightarrow \infty$.

We can now use Eqs. (168), (169) and P1, A2 to conclude that the evolution equation (20) for A^α assumes the form of Eqs. (149) and (150) with $k = 1$ and $l = 1 - \epsilon_E$. It follows from Proposition 1 that for any given $\epsilon_A > \epsilon_E$,

$$A^\alpha = \mathcal{O}(e^{(-1+\epsilon_A)t}) , \quad (170)$$

as $t \rightarrow \infty$.

At this stage we have two different epsilons in Eqs. (169) and (170), with $\epsilon_A > \epsilon_E$. Equation (169) is equally valid, however, if we replace ϵ_E by ϵ_A , and in the interests of simplicity we choose to do this, dropping the subscripts on the epsilons. We will also make this ϵ -simplification in subsequent steps.

In a similar way, Eq. (22) leads to¹³

$$N^{\alpha\beta} = \mathcal{O}(e^{(-1+\epsilon)t}) . \quad (171)$$

Next, we derive intermediate results for Ω and r_α . For any given $\delta > 0$, P1 and A2 imply that¹⁴

$$-\frac{\gamma}{G_+} v^\alpha \boldsymbol{\partial}_\alpha \Omega \leq \delta \Omega , \quad (172)$$

for t sufficiently large. Then, for any given $\epsilon > 0$, Eqs. (23) and (168)–(172) give

$$\partial_t \Omega \leq \left[-\frac{\gamma}{G_+} (3 + v^2) + \epsilon \right] \Omega , \quad (173)$$

for t sufficiently large. The inequalities $1 \leq \gamma < 2$ imply that

$$-\frac{\gamma}{G_+} (3 + v^2) \leq -3 . \quad (174)$$

It follows from Eqs. (173) and (174) by integrating that

$$\Omega = \mathcal{O}(e^{(-3+\epsilon)t}) . \quad (175)$$

The constraint $(\mathcal{C}_\Lambda)_\alpha$, in conjunction with P1, A2 and Eq. (169), implies

$$r_\alpha = \mathcal{O}(e^{(-1+\epsilon)t}) . \quad (176)$$

With Eqs. (175) and (176), we now have enough information to derive an asymptotic expression for $\Sigma^{\alpha\beta}$. Indeed, Eq. (21) assumes the form of Eqs. (149) and (150) with $k = 3$ and $l = 2 - \epsilon$, which implies, using Proposition 1, that

$$\Sigma^{\alpha\beta} = \mathcal{O}(e^{(-2+\epsilon)t}) . \quad (177)$$

It follows from Eqs. (169)–(171), (176) and A2 that $\Omega_k = \mathcal{O}(e^{(-2+\epsilon)t})$, which in turn implies, using Eqs. (175), (177) and the constraint (\mathcal{C}_G) , that

$$\Omega_\Lambda - 1 = \mathcal{O}(e^{(-2+\epsilon)t}) . \quad (178)$$

¹³Here and in the future, we will omit the qualifier $t \rightarrow \infty$.

¹⁴Here we use A2 to write $\partial_i \Omega = \mathcal{O}(\Omega)$, and also use the fact that $E_\alpha^i \rightarrow 0$ to write $|E_\alpha^i| < \epsilon$, for t sufficiently large.

By using A2 and the constraint $(\mathcal{C}_\Lambda)_\alpha$, we can now conclude that

$$r_\alpha = \mathcal{O}(e^{(-3+\epsilon)t}) . \quad (179)$$

It remains to derive the decay rates of Ω and v^α . We now make use of the evolution equation for the scalar $v = \sqrt{v_\alpha v^\alpha}$, which can be derived from Eq. (24). The equation for v^2 is in fact given in Paper I; see Eq. (146). Using the preceding asymptotic expressions and A2, the equation for v assumes the form

$$\partial_t v = \frac{1}{G_-} (1 - v^2) (3\gamma - 4) v + g , \quad (180)$$

where $g = \mathcal{O}(e^{(-1+\epsilon)t})$. Equation (180) is in fact an asymptotically autonomous DE, a scalar version of the vector DE discussed by Horwood *et al* in Ref. [16]. We can thus use Theorem B.1 in Ref. [16], p. 15, to obtain the limiting behavior of v as $t \rightarrow \infty$. The domain D for v is the interval $0 < v < 1$, and the conditions H_1 and H_2 in Ref. [16] are satisfied. It is straightforward to verify that the solutions of the autonomous DE

$$\partial_t v = \frac{1}{G_-} (1 - v^2) (3\gamma - 4) v , \quad (181)$$

with initial conditions in D , satisfy

$$\lim_{t \rightarrow \infty} v = \begin{cases} 0 & \text{if } 1 \leq \gamma < \frac{4}{3}. \\ 1 & \text{if } \frac{4}{3} < \gamma < 2. \end{cases} \quad (182)$$

It then follows from Theorem B.1 in Ref. [16] that any solution of Eq. (180) with initial condition in D satisfies Eq. (182).

If $1 \leq \gamma < \frac{4}{3}$, Eq. (180) assumes the form of Eqs. (149) and (150) with $k = -(3\gamma - 4)$ and $l = 1 - \epsilon$. Proposition 1 then implies

$$v = \mathcal{O}(e^{(3\gamma-4+\epsilon)t}) . \quad (183)$$

If $\frac{4}{3} < \gamma < 2$, we write the evolution equation for v [cf. Paper I, Eq. (146)] in the form

$$\partial_t (1 - v^2) = -v^\alpha \partial_\alpha (1 - v^2) - \frac{2}{G_-} \left[(3\gamma - 4) v^2 + \mathcal{O}(e^{(-1+\epsilon)t}) \right] (1 - v^2) . \quad (184)$$

P1 and A2 imply that for any given $\delta > 0$,¹⁵

$$-v^\alpha \partial_\alpha (1 - v^2) \leq \delta (1 - v^2) . \quad (185)$$

Then Eq. (184) assumes the form of Eqs. (149) and (150) with $k = 2 \frac{(3\gamma-4)}{(2-\gamma)}$ and $B_{\alpha\beta} = 0$. Proposition 1 implies

$$1 - v^2 = \mathcal{O}(e^{[-2 \frac{(3\gamma-4)}{(2-\gamma)} + \epsilon]t}) . \quad (186)$$

For $\gamma = \frac{4}{3}$, Eq. (24) gives

$$\partial_t v^\alpha = \mathcal{O}(e^{(-1+\epsilon)t}) , \quad (187)$$

which implies

$$v^\alpha = \hat{v}^\alpha + \mathcal{O}(e^{(-1+\epsilon)t}) . \quad (188)$$

Using Eq. (182), the inequality (173) can be strengthened to read

$$\partial_t \Omega \leq \begin{cases} (-3\gamma + \epsilon) \Omega & \text{for } 1 \leq \gamma < \frac{4}{3}, \\ (-4 + \epsilon) \Omega & \text{for } \frac{4}{3} < \gamma < 2, \end{cases} \quad (189)$$

where the ϵ has been redefined. If $\gamma = \frac{4}{3}$, Eq. (173) simplifies directly to

$$\partial_t \Omega \leq (-4 + \epsilon) \Omega . \quad (190)$$

Inequalities (189) and (190) now give

$$\Omega = \begin{cases} \mathcal{O}(e^{(-3\gamma+\epsilon)t}) & \text{for } 1 \leq \gamma < \frac{4}{3}. \\ \mathcal{O}(e^{(-4+\epsilon)t}) & \text{for } \frac{4}{3} \leq \gamma < 2. \end{cases} \quad (191)$$

¹⁵Since $1 - v^2 \rightarrow 0$ in this case, we can use A2 to write $\partial_t (1 - v^2) = \mathcal{O}(1 - v^2)$, and proceed as in footnote 14.

Stage 2

In the second stage we use the asymptotic decay rates found in Stage 1 to successively write the evolution equations in a form to which Proposition 3 can be applied.

We begin by considering $E_\alpha{}^i$. The decay rates in Stage 1 and Eq. (12) gives

$$q = -1 + \mathcal{O}(e^{(-2+\epsilon)t}) . \quad (192)$$

The evolution equation (18) for $E_\alpha{}^i$ can now be written in the form

$$\partial_t E_\alpha{}^i = -E_\alpha{}^i + g_\alpha{}^i , \quad (193)$$

where $g_\alpha{}^i = \mathcal{O}(e^{(-3+\epsilon)t})$. This equation, in conjunction with Proposition 3, implies that

$$E_\alpha{}^i = \hat{E}_\alpha{}^i e^{-t} + \mathcal{O}(e^{(-3+\epsilon)t}) . \quad (194)$$

In a similar way the evolution equations (20) and (22) lead to

$$A^\alpha = \hat{A}^\alpha e^{-t} + \mathcal{O}(e^{(-3+\epsilon)t}) \quad (195)$$

$$N^{\alpha\beta} = \hat{N}^{\alpha\beta} e^{-t} + \mathcal{O}(e^{(-3+\epsilon)t}) . \quad (196)$$

We now consider the evolution equation (23) for Ω . For $1 \leq \gamma < \frac{4}{3}$, we obtain

$$\partial_t \Omega = -3\gamma \Omega + g , \quad (197)$$

where $g = \mathcal{O}(e^{(3\gamma-8+\epsilon)t})$, with Ωv^2 being the dominant term in g . For $\frac{4}{3} < \gamma < 2$, we obtain

$$\partial_t \Omega = -4\Omega + g , \quad (198)$$

where $g = \mathcal{O}(e^{(-5+\epsilon)t} + e^{(-4-2\frac{(3\gamma-4)}{(2-\gamma)}+\epsilon)t})$, with $\partial_\alpha \Omega$, $\Omega \partial_\alpha v^\alpha$, $(A_\alpha v^\alpha) \Omega$ and $\Omega(1-v^2)$ being the dominant terms in g . For $\gamma = \frac{4}{3}$, we obtain

$$\partial_t \Omega = -4\Omega + \mathcal{O}(e^{(-5+\epsilon)t}) . \quad (199)$$

Applying Proposition 3 yields

$$\Omega = \begin{cases} \hat{\Omega} e^{-3\gamma t} + \mathcal{O}(e^{(3\gamma-8+\epsilon)t}) & \text{for } 1 \leq \gamma < \frac{4}{3}. \\ \hat{\Omega} e^{-4t} + \mathcal{O}(e^{(-5+\epsilon)t} + e^{(-4-2\frac{(3\gamma-4)}{(2-\gamma)}+\epsilon)t}) & \text{for } \frac{4}{3} < \gamma < 2. \\ \hat{\Omega} e^{-4t} + \mathcal{O}(e^{(-5+\epsilon)t}) & \text{for } \gamma = \frac{4}{3}. \end{cases} \quad (200)$$

At this stage, the evolution equation for $\Sigma^{\alpha\beta}$ assumes the form

$$\partial_t \Sigma^{\alpha\beta} = -3\Sigma^{\alpha\beta} - 3\hat{\mathcal{S}}^{\alpha\beta} e^{-2t} + g^{\alpha\beta} , \quad (201)$$

where $g^{\alpha\beta} = \mathcal{O}(e^{(-4+\epsilon)t})$. By imitating the proof of Proposition 3, we conclude that

$$\Sigma^{\alpha\beta} = -3\hat{\mathcal{S}}^{\alpha\beta} e^{-2t} + \hat{\Sigma}^{\alpha\beta} e^{-3t} + \mathcal{O}(e^{(-4+\epsilon)t}) . \quad (202)$$

At this stage (\mathcal{C}_G) and A3 give

$$\Omega_\Lambda - 1 = -\hat{\Omega}_k e^{-2t} + \mathcal{O}(e^{-3\gamma t} + e^{-4t}) , \quad (203)$$

while $(\mathcal{C}_\Lambda)_\alpha$ and A3 lead to

$$r_\alpha = -\frac{1}{2}(\hat{E}_\alpha{}^i \partial_i \hat{\Omega}_k) e^{-3t} + \mathcal{O}(e^{-(1+3\gamma)t} + e^{-5t}) . \quad (204)$$

We now consider the evolution equation (24) for v^α . For $\gamma = 1$, we obtain

$$\partial_t v^\alpha = -v^\alpha + \mathcal{O}(e^{(-3+\epsilon)t}) . \quad (205)$$

Proposition 3 implies

$$v^\alpha = \hat{v}^\alpha e^{-t} + \mathcal{O}(e^{(-3+\epsilon)t}) . \quad (206)$$

For $1 < \gamma < \frac{4}{3}$, we obtain

$$\partial_t v^\alpha = (3\gamma - 4) v^\alpha + g^\alpha, \quad (207)$$

where $g^\alpha = \mathcal{O}(e^{-t} + e^{3(3\gamma-4+\epsilon)t})$, with $\hat{\boldsymbol{\theta}}_\alpha \ln \Omega$ and $v^2 v^\alpha$ being the dominant terms in g^α . Proposition 3 implies

$$v^\alpha = \hat{v}^\alpha e^{(3\gamma-4)t} + \mathcal{O}(e^{-t} + e^{3(3\gamma-4+\epsilon)t}). \quad (208)$$

For $\gamma = \frac{4}{3}$, we already have the result:

$$v^\alpha = \hat{v}^\alpha + \mathcal{O}(e^{(-1+\epsilon)t}). \quad (209)$$

For $\frac{4}{3} < \gamma < 2$, we obtain from Eq. (184)

$$\partial_t(1 - v^2) = -2 \frac{(3\gamma - 4)}{(2 - \gamma)} (1 - v^2) + g, \quad (210)$$

where $g = \mathcal{O}(e^{(-1-2\frac{(3\gamma-4)}{(2-\gamma)}+\epsilon)t} + e^{(-4\frac{(3\gamma-4)}{(2-\gamma)}+\epsilon)t})$, with $(1 - v^2) \hat{\boldsymbol{\theta}}_\alpha v^\alpha$, $(1 - v^2) A_\alpha v^\alpha$ and $(1 - v^2)^2$ being the dominant terms in g . Proposition 3 implies

$$1 - v^2 = (1 - \hat{v}^2) e^{-2\frac{(3\gamma-4)}{(2-\gamma)}t} + \mathcal{O}(e^{(-1-2\frac{(3\gamma-4)}{(2-\gamma)}+\epsilon)t} + e^{(-4\frac{(3\gamma-4)}{(2-\gamma)}+\epsilon)t}). \quad (211)$$

Using the results obtained in Stage 2, we can repeat Stage 2 to eliminate the epsilons in the \mathcal{O} -terms as follows. Equation (192) now reads $q = -1 + \mathcal{O}(e^{-2t})$, and the function g_α^i in Eq. (193) satisfies $g_\alpha^i = \mathcal{O}(e^{-3t})$; similarly for the other variables.

The position-dependent coefficients \hat{E}_α^i , \hat{A}^α and $\hat{N}_{\alpha\beta}$ are required to satisfy the constraints

$$0 = 2(\hat{E}_{[\alpha}^j \partial_j - \hat{A}_{[\alpha} \hat{E}_{\beta]}^i - \epsilon_{\alpha\beta\delta} \hat{N}^{\delta\gamma} \hat{E}_\gamma^i) \quad (212)$$

$$0 = \hat{E}_\beta^i \partial_i (\hat{N}^{\alpha\beta} + \epsilon^{\alpha\beta\gamma} \hat{A}_\gamma) - 2\hat{A}_\beta \hat{N}^{\alpha\beta}, \quad (213)$$

which arise, respectively, from $(\mathcal{C}_{\text{com}})^i{}_{\alpha\beta}$ and $(\mathcal{C}_J)^\alpha$, at order e^{-2t} . Equation (212) can be solved to express the coefficients \hat{A}^α and $\hat{N}_{\alpha\beta}$ in terms of the coefficients \hat{E}_α^i and their inverse $\hat{E}^{\alpha}{}_i$. These expressions, given by Eqs. (75) and (76), satisfy Eq. (213) identically. The constraint $(\mathcal{C}_C)^\alpha$ provides a restriction on \hat{v}^α , given by Eqs. (82) and (83). \square

C Derivation of asymptotically flat-FL past dynamics

In this appendix we derive the asymptotic expansions (118)–(123) for G_0 cosmologies that are past asymptotic to the flat FL solution in the sense of Eqs. (115)–(117). For ease of reference, we list the complete set of restrictions below.

A1: E_α^i , A^α , $N_{\alpha\beta}$, r_α , $\Sigma_{\alpha\beta}$, $1 - \Omega$, $\Omega_\Lambda \rightarrow 0$ as $t \rightarrow -\infty$, for fixed x^i .

A2: The partial derivatives of any dimensionless variable V are bounded as $t \rightarrow -\infty$. In addition, if V tends to zero, then $\partial_i V = \mathcal{O}(\|V\|)$ as $t \rightarrow -\infty$, for fixed x^i , where $\|V\|$ is the magnitude of V .

A3: If $V = \hat{V} e^{nt} + \mathcal{O}(e^{mt})$ as $t \rightarrow -\infty$, where $m > n > 0$, then $\partial_i V = \partial_i \hat{V} e^{nt} + \mathcal{O}(e^{mt})$ as $t \rightarrow -\infty$.

We organize the derivation into two stages, in which we repeatedly use the evolution equations, (10) and the constraints (\mathcal{C}_G) and $(\mathcal{C}_C)^\alpha$. In Stage 1, we obtain order estimates for all variables, primarily using Propositions 1 and 2 in App. A. In Stage 2, we use the results in Stage 1 in conjunction with Proposition 3 to obtain the explicit asymptotic expansions. The assumptions A2 and A3 are needed throughout in order to bound the spatial derivative terms in the evolution equations, in (10), and in the constraints (\mathcal{C}_G) and $(\mathcal{C}_C)^\alpha$.

Stage 1

First, A1, A2 and $(\mathcal{C}_C)^\alpha$ imply that $(\gamma/G_+) \Omega v^\alpha \rightarrow 0$. But we have $\Omega \rightarrow 1$. Thus we have

$$\lim_{t \rightarrow -\infty} v^\alpha = 0 . \quad (214)$$

It follows from A1, A2 and (12) that

$$\lim_{t \rightarrow -\infty} q = \frac{1}{2} (3\gamma - 2) . \quad (215)$$

We now introduce $\tau = -t$, for convenience in using Propositions 1–3.

We begin by considering E_α^i . Using A1 and Eq. (215), the evolution equation (18) for E_α^i assumes the form of Eqs. (149) and (150) with $k = \frac{1}{2} (3\gamma - 2)$ and $B_{\alpha\beta} = 0$. It follows from Proposition 1 that for any given $\epsilon_E > 0$,

$$E_\alpha^i = \mathcal{O}(e^{[-\frac{1}{2}(3\gamma-2)+\epsilon_E]\tau}) , \quad (216)$$

as $\tau \rightarrow \infty$.

We can now use Eqs. (215), (169) and A1, A2 to conclude that the evolution equation (20) for A^α assumes the form of Eqs. (149) and (150) with $k = \frac{1}{2} (3\gamma - 2)$ and $l = \frac{1}{2} (3\gamma - 2) - \epsilon_E$. It follows from Proposition 1 that for any given $\epsilon_A > \epsilon_E$,

$$A^\alpha = \mathcal{O}(e^{[-\frac{1}{2}(3\gamma-2)+\epsilon_A]\tau}) , \quad (217)$$

as $\tau \rightarrow \infty$. We make the ϵ -simplification as in Subsec. B.

In a similar way, Eqs. (22), (19) and (25) lead to¹⁶

$$N^{\alpha\beta} = \mathcal{O}(e^{[-\frac{1}{2}(3\gamma-2)+\epsilon]\tau}) \quad (218)$$

$$r_\alpha = \mathcal{O}(e^{[-\frac{1}{2}(3\gamma-2)+\epsilon]\tau}) \quad (219)$$

$$\Omega_\Lambda = \mathcal{O}(e^{(-3\gamma+\epsilon)\tau}) . \quad (220)$$

With Eqs. (216)–(219) and A2, the constraint $(\mathcal{C}_C)^\alpha$ gives

$$v^\alpha = \mathcal{O}(e^{[-\frac{1}{2}(3\gamma-2)+\epsilon]\tau}) . \quad (221)$$

Equation (21) now assumes the form of Eqs. (159) and (160) with $k = \frac{3}{2} (2 - \gamma)$ and $l = (3\gamma - 2) - \epsilon$. Since $\Sigma_{\alpha\beta}$ is bounded as $\tau \rightarrow \infty$ (it is assumed to tend to zero), Proposition 2 implies

$$\Sigma^{\alpha\beta} = \mathcal{O}(e^{[-(3\gamma-2)+\epsilon]\tau}) . \quad (222)$$

Lastly, the constraint (\mathcal{C}_G) implies

$$1 - \Omega = \mathcal{O}(e^{[-(3\gamma-2)+\epsilon]\tau}) . \quad (223)$$

Stage 2

In the second stage we use the asymptotic decay rates found in Stage 1 to successively write the evolution equations in a form to which Proposition 3 can be applied.

We begin by considering E_α^i . The decay rates in Stage 1 and Eq. (12) give

$$q = \frac{1}{2} (3\gamma - 2) + \mathcal{O}(e^{[-(3\gamma-2)+\epsilon]\tau}) . \quad (224)$$

The evolution equation (18) for E_α^i can now be written in the form

$$\partial_\tau E_\alpha^i = -\frac{1}{2} (3\gamma - 2) E_\alpha^i + g_\alpha^i , \quad (225)$$

¹⁶Here and in the future, we will omit the qualifier $\tau \rightarrow \infty$.

where $g_\alpha{}^i = \mathcal{O}(e^{[-\frac{3}{2}(3\gamma-2)+\epsilon]\tau})$. This equation, in conjunction with Proposition 3, implies that

$$E_\alpha{}^i = \hat{E}_\alpha{}^i e^{-\frac{1}{2}(3\gamma-2)\tau} + \mathcal{O}(e^{[-\frac{3}{2}(3\gamma-2)+\epsilon]\tau}) . \quad (226)$$

In a similar way the evolution equations (9), (25), (20), (22) and (19) lead to

$$H = \hat{H} e^{(3\gamma/2)\tau} + \mathcal{O}(e^{[(3\gamma/2)-(3\gamma-2)+\epsilon]\tau}) \quad (227)$$

$$\Omega_\Lambda = \hat{\Omega}_\Lambda e^{-3\gamma\tau} + \mathcal{O}(e^{[-3\gamma-(3\gamma-2)+\epsilon]\tau}) \quad (228)$$

$$A^\alpha = \hat{A}^\alpha e^{-\frac{1}{2}(3\gamma-2)\tau} + \mathcal{O}(e^{[-\frac{3}{2}(3\gamma-2)+\epsilon]\tau}) \quad (229)$$

$$N^{\alpha\beta} = \hat{N}^{\alpha\beta} e^{-\frac{1}{2}(3\gamma-2)\tau} + \mathcal{O}(e^{[-\frac{3}{2}(3\gamma-2)+\epsilon]\tau}) \quad (230)$$

$$r_\alpha = \hat{r}_\alpha e^{-\frac{1}{2}(3\gamma-2)\tau} + \mathcal{O}(e^{[-\frac{3}{2}(3\gamma-2)+\epsilon]\tau}) , \quad (231)$$

where $\hat{\Omega}_\Lambda = \Lambda/(3\hat{H}^2)$, and $\hat{r}_\alpha = -(\hat{E}_\alpha{}^i \partial_i \hat{H})/\hat{H}$ from (10).

At this stage, $(\mathcal{C}_C)^\alpha$ and A3 give

$$v^\alpha = -\frac{2}{3\gamma} \hat{r}^\alpha e^{-\frac{1}{2}(3\gamma-2)\tau} + \mathcal{O}(e^{[-\frac{3}{2}(3\gamma-2)+\epsilon]\tau}) , \quad (232)$$

while (\mathcal{C}_G) and A3 lead to

$$\Omega = 1 - \hat{\Omega}_k e^{-(3\gamma-2)\tau} + \mathcal{O}(e^{[-2(3\gamma-2)+\epsilon]\tau} + e^{-3\gamma\tau}) . \quad (233)$$

We now use the freedom to re-define the 1-parameter family of spacelike 3-surfaces $\mathcal{S}:\{t = \text{constant}\}$ while preserving the separable volume gauge¹⁷ to set $\hat{H} = (2/3\gamma) T_0^{-1}$, with T_0 a positive real-valued constant. As a result,

$$\hat{r}_\alpha = 0 , \quad (234)$$

and $v^\alpha = \mathcal{O}(e^{[-\frac{3}{2}(3\gamma-2)+\epsilon]\tau})$.

The evolution equation (21) for $\Sigma^{\alpha\beta}$ assumes the form

$$\partial_\tau \Sigma^{\alpha\beta} = \frac{3}{2} (2 - \gamma) \Sigma^{\alpha\beta} + 3\hat{\mathcal{S}}^{\alpha\beta} e^{-(3\gamma-2)\tau} + g^{\alpha\beta} , \quad (235)$$

where $g^{\alpha\beta} = \mathcal{O}(e^{[-2(3\gamma-2)+\epsilon]\tau})$. We rewrite Eq. (235) as

$$\partial_\tau \left(e^{-\frac{3}{2}(2-\gamma)\tau} \Sigma^{\alpha\beta} \right) = 3\hat{\mathcal{S}}^{\alpha\beta} e^{-(1+\frac{3}{2}\gamma)\tau} + g^{\alpha\beta} e^{-\frac{3}{2}(2-\gamma)\tau} , \quad (236)$$

and integrate from τ to ∞ to obtain

$$\Sigma^{\alpha\beta} = -\frac{6}{(3\gamma+2)} \hat{\mathcal{S}}^{\alpha\beta} e^{-(3\gamma-2)\tau} + \mathcal{O}(e^{[-2(3\gamma-2)+\epsilon]\tau}) . \quad (237)$$

Lastly, we derive the leading order coefficients for r_α and v^α . First,

$$q = \frac{1}{2} (3\gamma - 2) - \frac{1}{2} (3\gamma - 2) \hat{\Omega}_k e^{-(3\gamma-2)\tau} + \mathcal{O}(e^{[-2(3\gamma-2)+\epsilon]\tau} + e^{-3\gamma\tau}) . \quad (238)$$

Evolution equation (9) for H then yields

$$H = \hat{H} e^{(3\gamma/2)\tau} \left[1 + \frac{1}{2} \hat{\Omega}_k e^{-(3\gamma-2)\tau} + \mathcal{O}(e^{[-2(3\gamma-2)+\epsilon]\tau} + e^{-3\gamma\tau}) \right] . \quad (239)$$

Eq. (10) and the constraint $(\mathcal{C}_C)^\alpha$ then successively give

$$r_\alpha = e^{-\frac{3}{2}(3\gamma-2)\tau} \left[-\frac{1}{2} (\hat{E}_\alpha{}^i \partial_i \hat{\Omega}_k) + \mathcal{O}(e^{[-2(3\gamma-2)+\epsilon]\tau} + e^{-3\gamma\tau}) \right] , \quad (240)$$

¹⁷A transformation of the local coordinates of the form $\tilde{t} = t + \varphi(x^i) + \mathcal{O}(e^{(3\gamma-2)t})$, $\tilde{x}^i = x^i + \frac{1}{(3\gamma-2)} \hat{H}^{-2} \hat{g}^{ij} \partial_j \varphi e^{(3\gamma-2)t} + \mathcal{O}(e^{2(3\gamma-2)t} + e^{3\gamma t})$ will preserve the separable volume gauge.

and

$$v^\alpha = e^{-\frac{3}{2}(3\gamma-2)\tau} \left[\hat{v}^\alpha + \mathcal{O}(e^{[-2(3\gamma-2)+\epsilon]\tau} + e^{-3\gamma\tau}) \right], \quad (241)$$

where

$$\hat{v}^\alpha = \frac{1}{3\gamma} \delta^{\alpha\beta} (\hat{E}_\beta{}^i \partial_i \hat{\Omega}_k) + \frac{2}{\gamma(3\gamma+2)} \left(\hat{E}_\beta{}^i \partial_i \hat{S}^{\alpha\beta} - 3\hat{A}_\beta \hat{S}^{\alpha\beta} - \epsilon^{\alpha\beta\gamma} \hat{N}_{\beta\delta} \hat{S}_\gamma{}^\delta \right). \quad (242)$$

The twice-contracted 3-Bianchi identity (34) gives

$$0 = \hat{E}_\beta{}^i \partial_i \hat{S}^{\alpha\beta} - 3\hat{A}_\beta \hat{S}^{\alpha\beta} - \epsilon^{\alpha\beta\gamma} \hat{N}_{\beta\delta} \hat{S}_\gamma{}^\delta + \frac{1}{3} \delta^{\alpha\beta} (\hat{E}_\beta{}^i \partial_i \hat{\Omega}_k). \quad (243)$$

This simplifies \hat{v}^α to

$$\hat{v}^\alpha = \frac{1}{(3\gamma+2)} \delta^{\alpha\beta} (\hat{E}_\beta{}^i \partial_i \hat{\Omega}_k). \quad (244)$$

Using the results obtained in Stage 2, we can repeat Stage 2 to eliminate the epsilons in the \mathcal{O} -terms, as in App. B.

As in App. B, the position-dependent coefficients $\hat{E}_\alpha{}^i$, \hat{A}^α and $\hat{N}_{\alpha\beta}$ are required to satisfy the constraints (212) and (213), which arise, respectively, from $(\mathcal{C}_{\text{com}})^\alpha{}_\beta$ and $(\mathcal{C}_J)^\alpha$, at order $e^{-(3\gamma-2)\tau}$. Equation (212) can be solved to express the coefficients \hat{A}^α and $\hat{N}_{\alpha\beta}$ in terms of the coefficients $\hat{E}_\alpha{}^i$ and their inverse $\hat{E}^{\alpha}{}_i$. These expressions, given by Eqs. (75) and (76), satisfy Eq. (213) identically. \square

D Asymptotic expansions for event horizons

Equation (44) and expansion (68) imply that $t_\alpha = \mathcal{O}(e^{-t})$ as $t \rightarrow \infty$, subject to the assumption that \mathcal{E} satisfies A2. Equations (43) and (42) successively imply that

$$s = \mathcal{O}(e^{-t}), \quad \mathcal{E} = \mathcal{O}(e^{-t}). \quad (245)$$

With these results, the evolution equation (48) for K^α yields $\partial_t K^\alpha = \mathcal{O}(e^{-t})$, which can be integrated to give

$$K^\alpha = \hat{K}^\alpha + \mathcal{O}(e^{-t}). \quad (246)$$

With expansions (68) and (246), Eq. (51) is integrated to give

$$x^i(t) = x_\infty^i + \mathcal{O}(e^{-t}). \quad (247)$$

The coefficients $\hat{E}_\alpha{}^i[x^j(t)]$ are then Taylor-expanded:

$$\hat{E}_\alpha{}^i[x^j(t)] = \hat{E}_\alpha{}^i(x_\infty^j) + \mathcal{O}(e^{-t}). \quad (248)$$

With expansions (68), (246) and (248), Eq. (51) is integrated to give an improved expansion

$$x^i(t) = x_\infty^i - \hat{K}^\alpha \hat{E}_\alpha{}^i(x_\infty^j) e^{-t} + \mathcal{O}(e^{-2t}). \quad (249)$$

E Asymptotic expansions for particle horizons

Equation (44) and expansion (118) imply that $t_\alpha = \mathcal{O}(\eta)$ as $\eta \rightarrow 0$, subject to the assumption that $1/\mathcal{E}$ satisfies A2. Equations (43) and (42) successively imply that

$$s = \mathcal{O}(\eta), \quad 1/\mathcal{E} = \mathcal{O}(\eta^{2/(3\gamma-2)}). \quad (250)$$

With these results the evolution equation (48) for K^α yields

$$K^\alpha = \hat{K}^\alpha + \mathcal{O}(\eta). \quad (251)$$

With expansions (118) and (251), Eq. (51) is integrated to give

$$x^i(\eta) = x_{\text{BB}}^i + \mathcal{O}(\eta^2). \quad (252)$$

The coefficients $\hat{E}_\alpha{}^i[x^j(\eta)]$ are then Taylor-expanded:

$$\hat{E}_\alpha{}^i[x^j(\eta)] = \hat{E}_\alpha{}^i(x_{\text{BB}}^j) + \mathcal{O}(\eta). \quad (253)$$

With expansions (118), (251) and (253), Eq. (51) is integrated to give an improved expansion

$$x^i(\eta) = x_{\text{BB}}^i + \frac{2}{(3\gamma-2)} \hat{K}^\alpha \hat{E}_\alpha{}^i(x_{\text{BB}}^j) \eta + \mathcal{O}(\eta^2). \quad (254)$$

F Fluid kinematical variables via boost transformations

Let $\tilde{\mathbf{u}}$ denote a fluid 4-velocity vector field. Then fluid kinematical variables arise in the irreducible decomposition of the spacetime gradient of $\tilde{\mathbf{u}}$, i.e.,

$$\nabla_a \tilde{u}_b = -\tilde{u}_a \dot{\tilde{u}}_b + \tilde{H} \tilde{h}_{ab} + \tilde{\sigma}_{ab} + \tilde{\omega}_{ab} , \quad (255)$$

defining the fluid acceleration $\dot{\tilde{u}}^a$, the fluid Hubble scalar \tilde{H} , the fluid shear rate $\tilde{\sigma}_{ab}$, and the fluid vorticity $\tilde{\omega}_{ab}$, respectively.

With respect to an Eulerian reference frame, comoving with a unit timelike congruence \mathbf{u} , we have

$$\tilde{u}^a = \Gamma (u^a + v^a) , \quad \tilde{h}^a_b = h^a_b + \Gamma^2 (v^2 u^a + v^a) u_b + \Gamma^2 (u^a + v^a) v_b , \quad (256)$$

where $h^a_b := \delta^a_b + u^a u_b$, and the Lorentz factor is defined by $\Gamma := 1/\sqrt{1-v^2}$, with $v^2 := v_a v^a$. We define the peculiar kinematical variables

$$\hat{\theta}(\mathbf{v}) := D_a v^a \quad \hat{\sigma}_{ab}(\mathbf{v}) := D_{(a} v_{b)} \quad \hat{\omega}^a(\mathbf{v}) := \epsilon^{abc} D_b v_c . \quad (257)$$

In the case when \mathbf{u} is vorticity-free [i.e., $\omega^a(\mathbf{u}) \equiv 0$], we find the boost transformations:

Fluid acceleration:

$$\begin{aligned} h^a_b \dot{\tilde{u}}^b &= \Gamma^2 [\dot{u}^a + \dot{v}^{(a)} + (H + \frac{1}{3} \hat{\theta}) v^a + (\sigma^a_b + \hat{\sigma}^a_b) v^b + \epsilon^{abc} \hat{\omega}_b v_c] \\ &\quad + \Gamma^4 (\dot{v}_b v^b + \frac{1}{3} \hat{\theta} v^2 + \hat{\sigma}_{bc} v^b v^c) v^a \end{aligned} \quad (258)$$

$$u_a \dot{\tilde{u}}^a = -\dot{\tilde{u}}_a v^a . \quad (259)$$

Fluid Hubble scalar:

$$\tilde{H} = \Gamma (H + \frac{1}{3} \hat{\theta} + \frac{1}{3} \dot{u}_a v^a) + \frac{1}{3} \Gamma^3 (\dot{v}_a v^a + \frac{1}{3} \hat{\theta} v^2 + \hat{\sigma}_{ab} v^a v^b) . \quad (260)$$

Fluid shear rate:

$$\begin{aligned} h^c_a h^d_b \tilde{\sigma}_{cd} &= \Gamma (\sigma_{ab} + \hat{\sigma}_{ab}) + \Gamma^3 v_{(a} (\dot{u}_{b)} + \dot{v}_{(b)}) + \sigma_{b)c} v^c + 2\Gamma^3 v_{(a} \hat{\sigma}_{b)c} v^c \\ &\quad - \frac{1}{3} \Gamma (\dot{u}_c v^c) [h_{ab} + \Gamma^2 v_a v_b] - \frac{1}{3} \Gamma^3 (\dot{v}_c v^c) [h_{ab} - 2\Gamma^2 v_a v_b] \\ &\quad + \frac{1}{3} \Gamma^5 \hat{\theta} v_a v_b - \frac{1}{9} \Gamma^3 \hat{\theta} v^2 [h_{ab} + \Gamma^2 v_a v_b] - \frac{1}{3} \Gamma^3 (\hat{\sigma}_{cd} v^c v^d) [h_{ab} - 2\Gamma^2 v_a v_b] \end{aligned} \quad (261)$$

$$u^a u^b \tilde{\sigma}_{ab} = \tilde{\sigma}_{ab} v^a v^b \quad (262)$$

$$u^b h^c_a \tilde{\sigma}_{bc} = -\tilde{\sigma}_{bc} v^b h^c_a . \quad (263)$$

Fluid vorticity:

$$h^c_a h^d_b \tilde{\omega}_{cd} = \Gamma \epsilon_{abc} \hat{\omega}^c + \Gamma^3 v_{[a} (\dot{u}_{b]} + \dot{v}_{[b]}) + \sigma_{b]c} v^c + 2\Gamma^3 v_{[a} \epsilon_{b]cd} \hat{\omega}^c v^d \quad (264)$$

$$u^b h^c_a \tilde{\omega}_{bc} = -\tilde{\omega}_{bc} v^b h^c_a . \quad (265)$$

Equations (258), (260), (261) and (264) provide the generalizations to G_0 cosmologies of Eqs. (1.15), (1.16), (1.27) and (1.26) given by King and Ellis [18] for the SH subcase.

The covariant derivatives of \mathbf{v} that appear in Eqs. (258)–(265) convert into orthonormal frame expressions according to

$$\dot{v}^{(a)} \longrightarrow \mathbf{e}_0(v^\alpha) - \epsilon^{\alpha\beta\gamma} \Omega_\beta v_\gamma \quad (266)$$

$$D_a v^a \longrightarrow (\mathbf{e}_\alpha - 2a_\alpha) (v^\alpha) \quad (267)$$

$$D_{(a} v_{b)} \longrightarrow (\mathbf{e}_{(a} + a_{(a} (v_{b)}) - \epsilon_{\gamma\delta(a} n_{b)})^\gamma v^\delta \quad (268)$$

$$\epsilon^{abc} D_b v_c \longrightarrow \epsilon^{\alpha\beta\gamma} (\mathbf{e}_\beta - a_\beta) (v_\gamma) - n^\alpha_\beta v^\beta . \quad (269)$$

Finally, we define Hubble-normalized fluid kinematical scalars by

$$\dot{U}^2 := \frac{1}{3} \left(\frac{\dot{\tilde{u}}_a}{\tilde{H}} \right) \left(\frac{\dot{\tilde{u}}^a}{\tilde{H}} \right) = \frac{1}{3} (\dot{U}_a \dot{U}^a) \quad (270)$$

$$\tilde{\Sigma}^2 := \frac{1}{6} \left(\frac{\tilde{\sigma}_{ab}}{\tilde{H}} \right) \left(\frac{\tilde{\sigma}^{ab}}{\tilde{H}} \right) = \frac{1}{6} (\tilde{\Sigma}_{ab} \tilde{\Sigma}^{ab}) \quad (271)$$

$$\tilde{W}^2 := \frac{1}{6} \left(\frac{\tilde{\omega}_{ab}}{\tilde{H}} \right) \left(\frac{\tilde{\omega}^{ab}}{\tilde{H}} \right) = \frac{1}{6} (\tilde{W}_{ab} \tilde{W}^{ab}) . \quad (272)$$

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